Supplementary Material
Admissible, Similar Tests: A characterization.
Appendix C.

APPENDIX C: ADDITIONAL RESULTS

C.1. *Asymptotic validity of the test in Result 1*

The test in Result 1 was derived under the assumption that the rotated reduced-form OLS estimators $(S_n', T_n')'$ have the exact distribution:

$$Q^n_{\beta,\pi,\Sigma} \equiv \mathcal{N}_{2k}\left(\begin{matrix} ([b_0' \otimes \mathbb{I}_k)\Sigma(b_0 \otimes \mathbb{I}_k)]^{-1/2}(\beta-\beta_0)\sqrt{n}\pi \\ [(a_0' \otimes \mathbb{I}_k)\Sigma^{-1}(a_0 \otimes \mathbb{I}_k)]^{-1/2}(a_0' \otimes \mathbb{I}_k)\Sigma^{-1}(a \otimes \mathbb{I}_k)\sqrt{n}\pi \end{matrix}, \ \mathbb{I}_{2k} \right),$$

where Σ is of the form $\Psi \otimes \Phi$. In any finite sample, however, the law of $(S_n', T_n')'$ is a function of (β,π), the sample size, and the joint distribution between the instrumental variables and reduced-form residuals, denoted F. In fact, one can write:

$$\begin{pmatrix} ([b_0' \otimes \mathbb{I}_k)\widehat{\Sigma}(b_0 \otimes \mathbb{I}_k)]^{-1/2}(b_0' \otimes \mathbb{I}_k) \\ [(a_0' \otimes \mathbb{I}_k)\widehat{\Sigma}^{-1}(a_0 \otimes \mathbb{I}_k)]^{-1/2}(a_0' \otimes \mathbb{I}_k)\widehat{\Sigma}^{-1} \end{pmatrix} \sqrt{n}\widehat{\gamma}_n \sim P^n_{\beta,\pi,F},$$

where $\widehat{\Sigma}$ is an estimator of the variance of $\sqrt{n}\widehat{\gamma}_n$. This variance depends on F and such dependence is denoted $\Sigma(F)$. The estimator $\widehat{\Sigma}$ need not have the Kronecker form, even when $\Sigma(F)$ does.

If one assumes that for n large enough the distributions P^n and Q^n are 'close' to each other (under the null), then one would expect the rate of Type I error computed under P^n to be close to that obtained under Q^n.

Preliminaries: I introduce some additional notation in order to establish Part 2 of Result 1.

1. *Bounded Lipschitz Distance*: Let $d_{\text{BL}}(P,Q) = \sup_{h \in \text{BL}_1} |\mathbb{E}_P[h(X)] - \mathbb{E}_Q[h(X)]|$ denote the Bounded Lipschitz distance between any pair of probability measures P and Q. For definitions and notation, see p. 73, Section 1.12 of Van der Vaart and Wellner (1996). Note also that the Bounded Lipschitz metric is equivalent to the 'β' metric between Borel probability measures defined in p. 394 of Dudley (2002).

2. *δ-Expansion of a set A:* For any $\delta > 0$ let A^δ denote the δ-expansion of the set $A \subseteq \mathbb{R}^m$. This is $A^\delta = \{y \in \mathbb{R}^m \mid d(x,y) \leq \delta \text{ for some } x \in A\}$.

3. *A bound on the distance between probability measures:* One can show that for any measurable set A and any $\delta > 0$:

$$-Q((A^c)^\delta \backslash A^c) - \frac{1}{\delta} d_{\text{BL}_1}(P,Q) \leq P(A) - Q(A) \leq \frac{1}{\delta} d_{\text{BL}_1}(P,Q) + Q(A^\delta \backslash A), \quad \text{(C.1)}$$

where A^c is the complement of $A \subseteq \mathbb{R}^m$. I use the right-hand side of this inequality to establish the main result.

Assumption L0: Suppose that the class of distributions $\mathcal{F}$ is such that:

$$\lim_{n\to\infty} \sup_{(\pi,F) \in \mathbb{R}^k \times \mathcal{F}} d_{\text{BL}}\left(P^n_{\beta_0,\pi,F}, Q^n_{\beta_0,\pi,\Sigma(F)}\right) \to 0.$$

That is, the Bounded Lipschitz distance between the measures $P^n_{\beta,\pi,F}$ and $Q^n_{\beta,\pi,\Sigma(F)}$ coverges to zero as the sample size grows large (uniformly over π and F).

Asymptotic Validity of the Test in Result 1: If Assumption L0 holds and there are constants $0 < \underline{\lambda} < \overline{\lambda} < \infty$ such that the eigenvalues of $\Sigma(F)$ belong to an interval $[\underline{\lambda}, \overline{\lambda}]$ for any

$F \in \mathcal{F}$, then:

$$\limsup_{n\to\infty} \sup_{(\pi,F)\in\mathbb{R}^k\times\mathcal{F}} P^n_{\beta_0,\pi,F}\left(z_{\text{WAP}}(S,T) - c_{\text{WAP}}(T,\alpha) > 0\right) \leq \alpha.$$

This means that the rate of Type I error of the test in Result 1 is uniformly controlled over $(\pi, \mathcal{F}) \in \mathbb{R}^k \times \mathcal{F}$.

Consider the test statistic

$$z(S,T) \equiv S'S - T'T + 8\ln\left(I_0\left[(1/8)\sqrt{(S'S-T'T)^2 + 4(S'T)^2}\right]\right)$$

and let $c(T;\alpha)$ denote its conditional critical value. I would like to show that if Assumption L0 holds over the class $\mathcal{F}$, then:

$$\limsup_{n\to\infty} \sup_{(\pi,F)\in\mathbb{R}^k\times\mathcal{F}} P^n_{\beta_0,\pi,F}\left(z(S,T) - c(T;\alpha) \geq 0\right) \leq \alpha. \tag{C.2}$$

I establish the asymptotic validity of the test in Result 1 in 6 steps:

Step 0: Define

$$A \equiv \{(s',t')' \in \mathbb{R}^{2k} \mid z(s,t) - c(t;\alpha) \geq 0\}.$$

Note immediately that Equation (C.1) implies that for any sample size n and any $(\pi, F) \in \mathbb{R}^k \times \mathcal{F}$:

$$\begin{aligned} P^n_{\beta_0,\pi,\mathcal{F}}(A) &\leq Q^n_{\beta_0,\pi,\Sigma(F)}(A) + \frac{1}{\delta} d_{\text{BL}_1}(P^n_{\beta_0,\pi,F}, Q^n_{\beta_0,\pi,\Sigma(F)}) + Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A), \\ &= \alpha + \frac{1}{\delta} d_{\text{BL}_1}(P^n_{\beta_0,\pi,F}, Q^n_{\beta_0,\pi,\Sigma(F)}) + Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A). \end{aligned}$$

where the last equality follows by the definition of the conditional critical value $c(T;\alpha)$. Thus, in order to establish (C.2), I need to show that

$$\frac{1}{\delta} d_{\text{BL}_1}(P^n_{\beta_0,\pi,F}, Q^n_{\beta_0,\pi,\Sigma(F)}) + Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A)$$

can be made arbitrary small, uniformly over the values of (π, F). By the weak convergence assumption in Part 2 of Result 1, for any fixed δ there is $M_\epsilon(\delta) \in \mathbb{N}$ such that whenever $n \geq M_\epsilon(\delta)$ the term $\delta^{-1} d_{\text{BL}_1}(P^n_{\beta_0,\pi,F}, Q^n_{\beta_0,\pi,\Sigma(F)})$ can be made smaller than ϵ. Thus, I only need to establish the following result.

Goal: For every $\epsilon > 0$ there is δ_ϵ and N_ϵ such that for all $n \geq N_\epsilon$

$$\sup_{(\pi,F)\in\mathbb{R}^k\times F} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon} \backslash A) \leq \epsilon.$$

The proof of this result requires a series of intermediate steps. I exploit the fact that the test statistic $z(s,t)$ satisfies a Lipschitz condition whenever (s,t) is restricted to an appropriate set.

Step 1: (A bound on $Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A)$): Define the sets

$$B(\underline{b}_1, \overline{b}_2) \equiv \{(s',t')' \in \mathbb{R}^{2k} \mid s's \in [\underline{b}_1, \overline{b}_2]\ \}, \tag{C.3}$$

$$C(\underline{c}_1, \overline{c}_2) \equiv \{(s',t')' \in \mathbb{R}^{2k} \mid t't \in [\underline{c}_1, \overline{c}_2]\ \}, \tag{C.4}$$

$$D(\underline{d}_1, \overline{d}_2) \equiv \{(s',t')' \in \mathbb{R}^{2k} \mid (s't)^2/t't \in [\underline{d}_1, \overline{d}_2]\ \}, \tag{C.5}$$

where $\underline{b}_1, \overline{b}_2, \underline{c}_1, \overline{c}_2, \underline{d}_1, \overline{d}_2$ are positive, finite constants. I want to study the behavior of $A^\delta \backslash A$ inside and outside the sets defined above. Note that for any n, π, F and δ:

$$\begin{aligned}
Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A) &= Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A \cap [B(\underline{b}_1, \overline{b}_1) \cap C(\underline{c}_1, \overline{c}_2) \cap D(\underline{d}_1, \overline{d}_2)]) \\
&+ Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A \cap [B(\underline{b}_1, \overline{b}_1) \cap C(\underline{c}_1, \overline{c}_2) \cap D(\underline{d}_1, \overline{d}_2)]^c) \\
&\quad \text{(by the additivity property of probability measures)} \\
&\leq Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A \cap [B(\underline{b}_1, \overline{b}_1) \cap C(\underline{c}_1, \overline{c}_2) \cap D(\underline{d}_1, \overline{d}_2)]) \\
&+ Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A \cap B^c(\underline{b}_1, \overline{b}_1)) + Q^n_{\beta_0,\pi\Sigma(F)}(A^\delta \backslash A \cap C^c(\underline{c}_1, \overline{c}_2)) \\
&+ Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A \cap D(\underline{d}_1, \overline{d}_2)) \\
&\quad \text{(where I have used Boole's inequality)} \\
&\leq Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A \cap [B(\underline{b}_1, \overline{b}_1) \cap C(\underline{c}_1, \overline{c}_2) \cap D(\underline{d}_1, \overline{d}_2)]) \\
&+ Q^n_{\beta_0,\pi,\Sigma(F)}(B^c(\underline{b}_1, \overline{b}_1)) + Q^n_{\beta_0,\pi\Sigma(F)}(D^c(\underline{d}_1, \overline{d}_2)) \\
&+ Q^n_{\beta_0,\pi\Sigma(F)}(A^\delta \backslash A \cap C^c(\underline{c}_1, \overline{c}_2)) \\
&\quad \text{(by the monotonicity of probability measures)} \\
&\leq Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A \cap [B(\underline{b}_1, \overline{b}_1) \cap C(\underline{c}_1, \overline{c}_2) \cap D(\underline{d}_1, \overline{d}_2)]) \\
&+ \mathbb{P}(\chi^2_k \notin (\underline{b}_1, \overline{b}_2)) + \mathbb{P}(\chi^2_1 \notin (\overline{d}_1, \overline{d}_2)) \\
&+ Q^n_{\beta_0,\pi\Sigma(F)}(A^\delta \backslash A \cap C^c(\underline{c}_1, \overline{c}_2)),
\end{aligned}$$

where the second to last line has uses the fact that under any probability measure $Q^n_{\beta_0,\pi,\Sigma(F)}$:[14]

$$S_n' S_n \overset{Q^n_{\beta_0,\pi,F}}{\sim} \chi^2_k \text{ and } (S_n' T_n)^2 / T_n' T_n \overset{Q^n_{\beta_0,\pi,F}}{\sim} \chi^2_1.$$

<u>MAIN CONCLUSION OF STEP 1:</u> I have shown that for any $\delta > 0$ and any positive finite constants $\underline{b}_1, \overline{b}_2, \underline{c}_1, \overline{c}_2, \underline{d}_1, \overline{d}_2$:

$$\begin{aligned}
Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A) &\leq Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta \backslash A \cap [B(\underline{b}_1, \overline{b}_1) \cap C(\underline{c}_1, \overline{c}_2) \cap D(\underline{d}_1, \overline{d}_2)]) \\
&+ \mathbb{P}(\chi^2_k \notin (\underline{b}_1, \overline{b}_2)) + \mathbb{P}(\chi^2_1 \notin (\overline{d}_1, \overline{d}_2)) \\
&+ Q^n_{\beta_0,\pi\Sigma(F)}(A^\delta \backslash A \cap C^c(\underline{c}_1, \overline{c}_2)).
\end{aligned} \tag{C.6}$$

I now argue that for an appropriate selection of constants, the test statistic $z(s,t)$ and its critical value $c(t;\alpha)$ satisfy a Lipschitz condition when restricted to the set $[B(\underline{b}_1, \overline{b}_1) \cap C(\underline{c}_1, \overline{c}_2) \cap D(\underline{d}_1, \overline{d}_2)]$.

STEP 2—PART A): (Lipschitz property of $z(s,t)$): I show that there exists a constant M_1—that only depends on the sets B, C, D—such that for any

$$(s_0', t_0')', (s_1', t_1')' \in \mathcal{K} \equiv [B(\underline{b}_1, \overline{b}_1) \cap C(\underline{c}_1, \overline{c}_2) \cap D(\underline{d}_1, \overline{d}_2)],$$

then:

$$|z(s_0, t_0) - z(s_1, t_1)| < M_1 ||(s_0', t_0') - (s_1', t_1')||.$$

To verify the Lipschitz property on $[B(\underline{b}_1, \overline{b}_1) \cap C(\underline{c}_1(\epsilon), \overline{c}_2(\epsilon)) \cap D(\underline{d}_1, \overline{d}_2)]$, it is sufficient to show that, over this set, the derivative of $z(s,t)$ is continuous in its arguments. This observation, together with the fact that $[B(\underline{b}_1, \overline{b}_1) \cap C(\underline{c}_1(\epsilon), \overline{c}_2(\epsilon)) \cap D(\underline{d}_1, \overline{d}_2)]$ is compact gives the desired result. Note

[14] I also use the fact that $\Sigma(F)$ is invertible for any element $F \in \mathcal{F}$.

that the partial derivative of $z(s,t)$ with respect to s is given by:

$$z_s(s,t) = 2S + \frac{8I_1\left((1/8)\sqrt{(s's-t't)^2+4t't(s't)^2/t't}\right)}{I_0\left((1/8)\sqrt{(s's-t't)^2+4t't(s't)^2/t't}\right)}\frac{1}{8}\frac{4(s's-t't)s+8t't((s't)/t't)s}{2\sqrt{(s's-t't)^2+4t't((s't)^2/t't)}} \tag{C.7}$$

$$z_t(s,t) = 2t + \frac{8I_1\left((1/8)\sqrt{(s's-t't)^2+4t't(s't)^2/t't}\right)}{I_0\left((1/8)\sqrt{(s's-t't)^2+4t't(s't)^2/t't}\right)}\frac{1}{8}\frac{4(s's-t't)t+8t't((s't)/t't)t}{2\sqrt{(s's-t't)^2+4t't((s't)^2/t't)}} \tag{C.8}$$

where $I_v(\cdot)$ is the modified Bessel function of the first kind defined in Section 9.6, p. 374 of Abramowitz and Stegun (1964). The formulae above use the fact that the derivative of the modified Bessel function of the first kind of order 0, I_0, is the modified Bessel function of order 1, I_1; see formula 9.6.27 in p. 376 of Abramowitz and Stegun (1964). The continuity of the derivatives and the fact that:

$$\sqrt{(s's-t't)^2+4t't(s't)^2/t't}$$

is bounded away from zero over the set

$$[B(\underline{b}_1,\overline{b}_1)\cap C(\underline{c}_1,\overline{c}_2)\cap D(\underline{d}_1,\overline{d}_2)]$$

implies that the Lipschitz condition holds.

Step 2—Part b): (Lipschitz property of $c(t;\alpha)$): Part a) showed that for any selection of constants (b,c,d) the test statistic $z(s,t)$ satisfies the Lipschitz condition when restricted to $\mathcal{K}$. I now introduce a parameter γ and show that for any given $\gamma>0$, $\epsilon>0$ and any pair of constants $\underline{c}_1$, $\overline{c}_2$, one can find $\underline{b}_1^*$, $\overline{b}_2^*$, $\underline{d}_1^*$, $\overline{d}_2^*$, and M_2—that depend on $\underline{c}_1$, $\overline{c}_2$, γ and ϵ—such that for any t_0, t_1 satisfying:

$$(s_0,t_0),(s_1,t_1)\in\mathcal{K}^*\equiv[B(\underline{b}_1^*,\overline{b}_1^*)\cap C(\underline{c}_1,\overline{c}_2)\cap D(\underline{d}_1^*,\overline{d}_2^*)],\quad \text{for some } s_0,s_1\in\mathbb{R}^k$$

the critical value function satisfies a Lipschitz-type condition:

$$|c(t_0;\alpha)-c(t_1;\alpha)|<M_2||t_0-t_1||+\gamma/2,$$

and:

$$\mathbb{P}(\chi_k^2\notin(\underline{b}_1^*,\overline{b}_1^*)+\mathbb{P}(\chi_1^2\notin(\underline{d}_1^*,\overline{d}_2^*))<\epsilon/3.$$

The parameter γ controls how close to is the conditional critical value to satisfy the Lipschitz condition.

To show this, note that for any constant $z\in\mathbb{R}$ and $(\pi,\mathcal{F})\in\mathbb{R}^k\times\mathcal{F}$:

$$\begin{aligned}
Q^n_{\beta_0,\pi,\Sigma(F)}\left(z(s,t_0)\le z\mid t=t_0\right) &= \mathbb{P}\left(z(S,t_0)\le z\right),\ S\sim\mathcal{N}(0,\mathbb{I}_k)\\
&= \mathbb{P}\left(z(S,t_0)-z(S,t_1)+z(S,t_1)\le z\right)\\
&= \mathbb{P}\left(z(S,t_1)\le z-(z(S,t_0)-z(S,t_1))\text{ and }(S,t_0),(S,t_1)\in\mathcal{K}\right)\\
&+ \mathbb{P}\left(z(S,t_1)\le z-(z(S,t_0)-z(S,t_1))\text{ and }(S,t_0),(S,t_1)\in\mathcal{K}^c\right).
\end{aligned}$$

where $\mathcal{K}\equiv[B(\underline{b}_1,\overline{b}_1)\cap C(\underline{c}_1,\overline{c}_2)\cap D(\underline{d}_1,\overline{d}_2)]$. Since $z(s,t)$ satisfies the Lipschitz condition in $\mathcal{K}$ with constant $M_1(\mathcal{K})$ it follows that:

$$\begin{aligned}
\mathbb{P}\Big(z(S,t_1) \leq z - M_1(\mathcal{K})||t_0 - t_1||\Big) &\leq \mathbb{P}\Big(z(S,t_1) \leq z - M_1(\mathcal{K})||t_0 - t_1|| \text{ and } (S,t_0),(S,t_1) \in \mathcal{K}\Big) \\
&+ \mathbb{P}\Big(z(S,t_1) \leq z - M_1(\mathcal{K})||t_0 - t_1|| \text{ and } (S,t_0),(S,t_1) \in \mathcal{K}^c\Big) \\
&\leq \mathbb{P}\Big(z(S,t_1) \leq z - (z(S,t_0) - z(S,t_1)) \text{ and } (S,t_0),(S,t_1) \in \mathcal{K}\Big) \\
&+ \mathbb{P}((S,t_0),(S,t_1) \in \mathcal{K}^c),
\end{aligned}$$

which implies that:

$$\mathbb{P}\Big(z(S,t_1) \leq z - M_1(\mathcal{K})||t_0 - t_1||\Big) - \mathbb{P}((S,t_0),(S,t_1) \in \mathcal{K}^c)$$

is less than or equal to

$$\mathbb{P}\Big(z(S,t_1) \leq z - (z(S,t_0) - z(S,t_1)) \text{ and } (S,t_0),(S,t_1) \in \mathcal{K}\Big).$$

Note also that:

$$\mathbb{P}\Big(z(S,t_1) \leq z - (z(S,t_0) - z(S,t_1)) \text{ and } (S,t_0),(S,t_1) \in \mathcal{K}\Big) \leq \mathbb{P}\Big(z(S,t_1) \leq z + M_1(\mathcal{K})||t_0 - t_1||\Big).$$

Note now that for any $t \in \mathbb{R}^k$, the critical value function is continuous in α. Therefore, there exists a positive constant, $\eta_\gamma(\underline{c}_1, \overline{c}_2) > 0$, such that for any t such that $t't \in [\underline{c}_1, \overline{c}_2]$:

$$|c(t; \alpha + \eta_\gamma(\underline{c}_1, \overline{c}_2)) - c(t;\alpha)| \leq \gamma/2, \quad |c(t; \alpha - \eta_\gamma(\underline{c}_1, \overline{c}_2)) - c(t;\alpha)| \leq \gamma/2.$$

Since for any vectors $t_0, t_1 \neq \mathbf{0}_{k\times 1}$:

$$\mathbb{P}((S,t_0),(S,t_1) \in \mathcal{K}^c) \leq \mathbb{P}(\chi^2_k \notin (\underline{b}_1, \overline{b}_1) + \mathbb{P}(\chi^2_1 \notin (\underline{d}_1, \overline{d}_2)),$$

one can then choose $0 < \underline{b}_1^* \equiv \underline{b}_1(\eta_\gamma(\underline{c}_1, \overline{c}_2), \epsilon) < \overline{b}_1^* \equiv \overline{b}_1(\eta_\gamma(\underline{c}_1, \overline{c}_2), \epsilon) < \infty$, and $0 < \underline{d}_1^* \equiv \underline{d}_1(\eta_\gamma(\underline{c}_1, \overline{c}_2), \epsilon) < \overline{d}_2^* \equiv \overline{d}_2(\eta_\gamma(\underline{c}_1, \overline{c}_2), \epsilon) < \infty$ such that

$$\mathbb{P}(\chi^2_k \notin (\underline{b}_1^*, \overline{b}_1^*) + \mathbb{P}(\chi^2_1 \notin (\underline{d}_1^*, \overline{d}_2^*)) < \min\{\eta_\gamma(\underline{c}_1, \overline{c}_2), \epsilon/3\}$$

for any t_0, t_1. This implies that:

$$\text{(C.9)} \qquad \mathbb{P}\Big(z(S,t_1) \leq z - M_2(\underline{c}_1, \overline{c}_2, \gamma, \epsilon)||t_0 - t_1||\Big) - \eta_\gamma(\underline{c}_1, \overline{c}_2) \leq \mathbb{P}\left(z(S,t_0) \leq z\right)$$

$$\text{(C.10)} \qquad \mathbb{P}\left(z(S,t_0) \leq z\right) \leq \mathbb{P}\Big(z(S,t_1) \leq z + M_2(\underline{c}_1, \overline{c}_2, \gamma, \epsilon)||t_0 - t_1||\Big) + \eta_\gamma(\underline{c}_1, \overline{c}_2),$$

where

$$M_2(\underline{c}_1, \overline{c}_2, \gamma, \epsilon) \equiv M_1\Big(\underline{b}_1(\eta_\gamma(\underline{c}_1, \overline{c}_2), \epsilon), \overline{b}_1(\eta_\gamma(\underline{c}_1, \overline{c}_2), \epsilon), \underline{c}_1, \overline{c}_2, \underline{d}_1(\eta_\gamma(\underline{c}_1, \overline{c}_2), \epsilon), \overline{d}_2(\eta_\gamma(\underline{c}_1, \overline{c}_2), \epsilon)\Big).$$

For simplicity we write M_2 instead of $M_2(\underline{c}_1, \overline{c}_2, \gamma, \epsilon)$ whenever it is convenient.

Since (C.9) holds for any z, in particular it holds for $z = c(t_1;\alpha) + M_2||t_0 - t_1||$. Consequently:

$$\begin{aligned}
\mathbb{P}\left(z(S,t_0) \leq c(t_1;\alpha) + M_2||t_0 - t_1||\right) &\geq \mathbb{P}\left(z(S,t_1) \leq c(t_1;\alpha)\right) - \eta_\gamma(\underline{c}_1, \overline{c}_2) \\
&= 1 - \alpha - \eta_\gamma(\underline{c}_1, \overline{c}_2).
\end{aligned}$$

This implies that

$$\text{(C.11)} \qquad c(t_0; \alpha + \eta_\gamma(\underline{c}_1, \overline{c}_2)) \leq c(t_1;\alpha) + M_2||t_0 - t_1||.$$

Likewise, equation (C.10) holds for any z, in particular it holds for $z = c(t_1;\alpha) - M_2||t_0 - t_1||$. This implies that:

$$\begin{aligned}\mathbb{P}\left(z(S,t_0) \leq c(t_1;\alpha) - M_2||t_0-t_1||\right) &\leq \mathbb{P}\left(z(S,t_1)\leq c(t_1;\alpha)\right) + \eta_\gamma(\underline{c}_1,\overline{c}_2)\\ &= (1-\alpha)+\eta_\gamma(\underline{c}_1,\overline{c}_2).\end{aligned}$$

This implies that:

$$c(t_0;\alpha-\eta_\gamma(\underline{c}_1,\overline{c}_2)) \geq c(t_1;\alpha) - M_2||t_0-t_1||. \tag{C.12}$$

<u>Main conclusion of Step 2</u>: Finally, (C.11)-(C.12) and the definition of $\eta_\gamma(\underline{c}_1,\overline{c}_2)$ imply that for any $\gamma>0$, $\epsilon>0$ and any pair of constants $\underline{c}_1$, $\overline{c}_2$, one can find $\underline{b}_1^*, \overline{b}_2^*$, $\underline{d}_1^*$, $\overline{d}_2^*$, and M_2—that depend on $\underline{c}_1$, $\overline{c}_2$, γ and ϵ—such that for any:

$$(s_0,t_0),(s_1,t_1)\in\mathcal{K}^*\equiv[B(\underline{b}_1^*,\overline{b}_1^*)\cap C(\underline{c}_1,\overline{c}_2)\cap D(\underline{d}_1^*,\overline{d}_2^*)],$$

the critical value function satisfies the Lipschitz-type condition:

$$|c(t_0;\alpha)-c(t_1;\alpha)| < M_2||t_0-t_1|| + \gamma/2.$$

and:

$$\mathbb{P}(\chi^2_k\notin(\underline{b}_1^*,\overline{b}_1^*) + \mathbb{P}(\chi^2_1\notin(\underline{d}_1^*,\overline{d}_2^*)) < \epsilon/3.$$

Step 3: (Exploiting the Lipschitz property to manipulate $A^\delta\backslash A$) The constants in Step 2 depend on $\gamma>0, \epsilon>0$ and $\underline{c}_1,\overline{c}_2$. This step fixes $\gamma>0$ and shows how to choose an appropriate enlargement of the set A as a function of γ. The Lipschitz condition established at the end of Step 2 allows for a convenient upper 'bound' on the set:

$$A^{\delta(\gamma)}\backslash A\;\cap[B(\underline{b}_1^*,\overline{b}_1^*)\cap C(\underline{c}_1,\overline{c}_2)\cap D(\underline{d}_1^*,\overline{d}_2^*)]. \tag{C.13}$$

In particular, I show that for any $\gamma>0$, there exists $\delta(\gamma)$ such that:

$$(s',t')'\in A^{\delta(\gamma)}\backslash A\;\cap[B(\underline{b}_1^*,\overline{b}_1^*)\cap C(\underline{c}_1,\overline{c}_2)\cap D(\underline{d}_1^*,\overline{d}_2^*)]$$

implies that:

$$-\gamma\leq z(s,t)-c(t;\alpha)\leq 0.$$

This inclusion relation is convenient as it allows for the selection of the auxiliary parameter γ to make the probability of the set (C.13) uniformly small over (π,F).

To establish the desired result, note that $x\equiv(s',t')'\in A^\delta\backslash A$ implies that:

$$z(s,t)-c(t;\alpha)<0,\;\text{(as } x\equiv(s',t')'\notin A),$$

and also that, for any δ, there exists $x_0(\delta)\equiv(s'_{0,\delta},t'_{0,\delta})'\in A$ such that

$$d(x,x_0(\delta))\leq\delta.$$

Since the functions $s's$, $t't$, $(s't)^2/(t't)$ defining the set:

$$\mathcal{K}^*\equiv[B(\underline{b}_1^*,\overline{b}_1^*)\cap C(\underline{c}_1,\overline{c}_2)\cap D(\underline{d}_1^*,\overline{d}_2^*)],$$

are Lipschitz continuous when restricted to $\mathcal{K}^*$, there exists δ^* small enough for which the corresponding $x_0(\delta^*)$ belongs to the set $\mathcal{K}^*$. In this case we have that:

$$\begin{aligned} ||(z(s,t)-c(t;\alpha))-(z(s_{0,\delta^*},t_{0,\delta^*})-c(t_0;\alpha))|| &\leq ||z(s,t)-z(s_{0,\delta^*},t_{0,\delta^*})||+||c(t;\alpha)-c(t_{0,\delta^*};\alpha)||, \\ &\leq (M_1(\mathcal{K}^*)+M_2(\underline{c}_1,\overline{c}_2,\gamma))d(x,x_0(\delta^*))+\gamma/2, \\ &\quad \text{(where I have used Step 2 part a) and b)),} \\ &\leq (M_1+M_2)\delta^*+\gamma/2 \end{aligned}$$

Since $x \notin A$ and $x_0(\delta^*) \in A$ implies that

$$0 \geq (z(s,t)-c(t;\alpha)) \geq (z(s,t)-c(t;\alpha))-(z(s_{0,\delta^*},t_{0,\delta^*})-c(t_0;\alpha)) \geq -(M_1+M_2)\delta^*-\gamma/2$$

<u>Main conclusion of Step 3</u>: Taking $\delta(\gamma) \equiv \min\{\delta^*, \frac{\gamma}{2(M_1+M_2)}\}$ it follows that

$$(s',t')' \in A^{\delta(\gamma)}\backslash A \cap [B(\underline{b}_1^*,\overline{b}_1^*) \cap C(\underline{c}_1,\overline{c}_2) \cap D(\underline{d}_1^*,\overline{d}_2^*)]$$

implies that:

$$-\gamma \leq z(s,t)-c(t;\alpha) \leq 0.$$

I now exploit this relation to show that one can choose γ to guarantee that

$$\sup_{\pi,F\in\mathbb{R}^k\times\mathcal{F}} Q^n_{\beta_0,\pi,\mathcal{F}}(A^{\delta(\gamma)}\backslash A \cap [B(\underline{b}_1^*,\overline{b}_1^*) \cap C(\underline{c}_1,\overline{c}_2) \cap D(\underline{d}_1^*,\overline{d}_2^*)])$$

can be made arbitrarily small.

Step 4: (Choosing γ as a function of ϵ) Remember that equation (C.6) in Step 1 established that for any $\delta > 0$ and any constants $\underline{b}_1,\overline{b}_1,\underline{c}_1,\overline{c}_1,\underline{d}_1,\overline{d}_1$:

$$\begin{aligned} Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta\backslash A) &\leq Q^n_{\beta_0,\pi,\Sigma(F)}(A^\delta\backslash A \cap [B(\underline{b}_1,\overline{b}_1) \cap C(\underline{c}_1,\overline{c}_2) \cap D(\underline{d}_1,\overline{d}_2)]) \\ &+ \mathbb{P}(\chi^2_k \notin (\underline{b}_1,\overline{b}_2)) + \mathbb{P}(\chi^2_1 \notin (\overline{d}_1,\overline{d}_2)) \\ &+ Q^n_{\beta_0,\pi\Sigma(F)}(A^\delta\backslash A \cap C^c(\underline{c}_1,\overline{c}_2)). \end{aligned}$$

Step 4 showed that for any $\gamma > 0$ there is way of selecting the enlargement parameter $\delta(\gamma) > 0$ and constants $\underline{b}_1^*,\overline{b}_2^*$, $\underline{d}_1^*$, $\overline{d}_2^*$—that depend on $\underline{c}_1$, $\overline{c}_2$ and γ—such tha the probabilityt:

$$Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta(\gamma)}\backslash A \cap [B(\underline{b}_1^*,\overline{b}_1^*) \cap C(\underline{c}_1,\overline{c}_2) \cap D(\underline{d}_1^*,\overline{d}_2^*)])$$

is less than or equal to

$$Q^n_{\beta_0,\pi,\Sigma(F)}(-\gamma \leq z(s,t)-c(t;\alpha) \leq 0 \cap [B(\underline{b}_1^*,\overline{b}_1^*) \cap C(\underline{c}_1,\overline{c}_2) \cap D(\underline{d}_1^*,\overline{d}_2^*)]) \tag{C.14}$$

I now show that there exists $\gamma_\epsilon > 0$ small enough such that:

$$Q^n_{\beta_0,\pi,\Sigma(F)}(-\gamma_\epsilon \leq z(s,t)-c(t;\alpha) \leq 0 \cap [B(\underline{b}_1^*,\overline{b}_1^*) \cap C(\underline{c}_1,\overline{c}_2) \cap D(\underline{d}_1^*,\overline{d}_2^*)]) < \epsilon/3$$

for any n,π,F.

To show this, define—for any t such that $t't \in [\underline{c}_1,\overline{c}_2]$—the function $\gamma_\epsilon(t)$ to satisfy:

$$\mathbb{P}_S(-\gamma_\epsilon(t) \leq g(S,t) \leq 0 \text{ and } S'S \in (\underline{b}_1^*,\overline{b}_1^*) \text{ and } \text{ and } (S't)^2/(t't) \in (\underline{d}_1^*,\overline{d}_1^*)), \quad S \sim \mathcal{N}(0,\mathbb{I}_k),$$

where $g(s,t;\alpha) \equiv z(s,t)-c(t;\alpha)$ and t is treated as fixed vector. Let

$$\gamma_\epsilon \equiv \inf_{\{t \,|\, t't\in[\underline{c}_1,\overline{c}_2]\}} \gamma_\epsilon(t)$$

and note that $\gamma_\epsilon > 0$ (otherwise, there will be a value t^* for which the distribution of $\mathbb{P}_S(g(S,t^*) =$

0) $> \epsilon/3$). Note that for any n, π, F:

$$Q^n_{\beta_0,\pi,\Sigma(F)}(-\gamma_\epsilon \leq g(s,t;\alpha) \leq 0 \text{ and } s's \in (\underline{b}_1^*, \overline{b}_1^*) \text{ and } t't \in (\underline{c}_1, \overline{c}_2), \text{ and } (s't)^2/t't \in (\underline{d}_1^*, \overline{d}_1^*)),$$

is the same as:

$$\int_{t \in C(\underline{c}_1, \overline{c}_2)} \Big(Q^n_{\beta_0,\pi,\Sigma(F)}(-\gamma_\epsilon \leq g(s,t;\alpha) \leq 0 \text{ and } s's \in (\underline{b}_1^*, \overline{b}_1^*) \text{ and } \text{ and } (s't)^2/t't \in (\underline{d}_1^*, \overline{d}_1^*) \mid t) \Big) d\mathbb{P}^n_{\beta_0,\pi,F}(t),$$

where $\mathbb{P}^n_{\beta_0,\pi,F}$ is the marginal distribution that $Q^n_{\beta_0,\pi,F}$ induces over (t). Note that (C.15) equals:

$$\int_{t \in C(\underline{c}_1, \overline{c}_2)} \Big(\mathbb{P}_S(-\gamma_\epsilon \leq g(S,t) \leq 0 \text{ and } S'S \in (\underline{b}_1^*, \overline{b}_1^*) \text{ and } \text{ and } (S't)^2/(t't) \in (\underline{d}_1^*, \overline{d}_1^*)) \Big) d\mathbb{P}^n_{\beta_0,\pi,F}(t),$$

$s|t$ has distribution $\mathcal{N}_k(\mathbf{0}, \mathbb{I}_k)$ for any n, π, F. And this is smaller than or equal:

$$\begin{aligned} &\int_{t \in C(\underline{c}_1, \overline{c}_2)} \Big(\mathbb{P}_S(-\gamma_\epsilon(t) \leq g(S,t) \leq 0 \text{ and } S'S \in (\underline{b}_1^*, \overline{b}_1^*) \text{ and } \text{ and } (S't)^2/(t't) \in (\underline{d}_1^*, \overline{d}_1^*)) \Big) d\mathbb{P}^n_{\beta_0,\pi,F}(t), \\ = \;& \mathbb{P}^n_{\beta_0,\pi,F}(t \in C(\underline{c}_1, \overline{c}_2)) \frac{\epsilon}{3} < \frac{\epsilon}{3}, \text{ (by definition of } \gamma_\epsilon(t)). \end{aligned}$$

<u>MAIN CONCLUSION OF STEP 4</u>: This means that for any $\epsilon > 0$ there exists $\gamma_\epsilon > 0$ small enough such that:

$$Q^n_{\beta_0,\pi,\Sigma(F)}(-\gamma_\epsilon \leq z(s,t) - c(t;\alpha) \leq 0 \cap [B(\underline{b}_1^*, \overline{b}_1^*) \cap C(\underline{c}_1, \overline{c}_2) \cap D(\underline{d}_1^*, \overline{d}_2^*)]) < \epsilon/3$$

for any n, π, F.

STEP 5 (CHOOSING $\underline{c}_1$ AND $\underline{c}_2$): Step 1 through Step 4 have shown that for any $\epsilon > 0$ there is a constant $\delta_\epsilon \equiv \delta(\gamma_\epsilon)$ and constants $\underline{b}_1^*, \overline{b}_2^*$, $\underline{d}_1^*$, $\overline{d}_2^*$—that depend on $\underline{c}_1$, $\overline{c}_2$ and ϵ such that for any n, π, F:

$$\begin{aligned} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon} \backslash A) \;&\leq\; Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon} \backslash A \,\cap [B(\underline{b}_1^*, \overline{b}_1^*) \cap C(\underline{c}_1, \overline{c}_2) \cap D(\underline{d}_1^*, \overline{d}_2^*)]) \\ &+\; \mathbb{P}(\chi^2_k \notin (\underline{b}_1^*, \overline{b}_2^*)) + \mathbb{P}(\chi^2_1 \notin (\overline{d}_1^*, \overline{d}_2^*)) \\ &+\; Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon} \backslash A \,\cap C^c(\underline{c}_1, \overline{c}_2)). \\ &\leq\; \frac{2\epsilon}{3} + Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta} \backslash A \,\cap C^c(\underline{c}_1, \overline{c}_2)). \end{aligned}$$

This means that:

$$\sup_{(\pi,F) \in \mathbb{R}^k \times \mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon} \backslash A) \leq \frac{2\epsilon}{3} + \sup_{(\pi,F) \in \mathbb{R}^k \times \mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon} \backslash A \,\cap C^c(\underline{c}_1, \overline{c}_2)).$$

Thus, I only need to show is that $\underline{c}_1$ and $\overline{c}_2$ can be chosen to make the second term on the right of the inequality above smaller than $\epsilon/3$. Let λ^* be defined as:

$$\lambda^* \equiv \max_{F \in \mathcal{F}} (a_0' \otimes \mathbb{I}_k) \Sigma(F)^{-1} (a_0 \otimes \mathbb{I}_k)$$

By assumption, there are constants $0 < \underline{\lambda} < \overline{\lambda} < \infty$ such that $\underline{\lambda} < \lambda^* < \overline{\lambda}$. Fix $c^* \in \mathbb{R}^k$ and partition $\mathbb{R}^k$ as follows:

$$\{\pi \in \mathbb{R}^k \mid : n||\pi||^2 \lambda^* \leq c^*\} \cup \{\pi \in \mathbb{R}^k \mid : n||\pi||^2 \lambda^* > c^*\} \equiv \Pi_1^n(c^*) \cup \Pi_2^n(c^*).$$

Note that

$$\sup_{(\pi,F)\in\mathbb{R}^k\times\mathcal{F}} Q^n_{\beta_0,\pi\Sigma(F)}(A^{\delta_\epsilon}\backslash A\ \cap C^c(\underline{c}_1,\overline{c}_2))$$

is smaller than or equal than the sum of:

$$\sup_{(\pi,F)\in\Pi^n_1(c^*)\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon}\backslash A\ \cap C^c(\underline{c}_1,\overline{c}_2)), \tag{C.15}$$

and

$$\sup_{(\pi,F)\in\Pi^n_2(c^*)\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon}\backslash A\ \cap C^c(\underline{c}_1,\overline{c}_2)) \tag{C.16}$$

Step 5—Part a): First, I bound the term (C.15). Let $\chi^2_k(c)$ denote a non-central chi-square with k degrees of freedom and centrality parameter c. Note that:

$$\begin{aligned}
\sup_{(\pi,F)\in\Pi^n_1(c^*)\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon}\backslash A\ \cap C^c(\underline{c}_1,\overline{c}_2)) &\leq \sup_{(\pi,F)\in\Pi^n_1(c^*)\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(C^c(\underline{c}_1,\overline{c}_2)),\\
&= \sup_{(\pi,F)\in\Pi^n_1(c^*)\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(t't\notin(\underline{c}_1,\overline{c}_2)),\\
&= \sup_{(\pi,F)\in\Pi^n_1(c^*)\times\mathcal{F}} \mathbb{P}\Big(\chi^2_k(n||\pi||^2\lambda(F))\notin(\underline{c}_1,\overline{c}_2)\Big),\\
&\quad (\text{where } \lambda(F)\equiv(a_0'\otimes\mathbb{I}_k)\Sigma(F)^{-1}(a_0\otimes\mathbb{I}_k)).
\end{aligned}$$

Therefore, one can choose constants $\underline{c}^*_1,\overline{c}^*_2$ that depend on c^* and ϵ (but do not depend on the sample size) such that for any $\underline{c}_1<\underline{c}^*_1$ and $\overline{c}_2>\overline{c}^*_2$:

$$\sup_{(\pi,F)\in\Pi^n_1(c^*)\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon}\backslash A\ \cap C^c(\underline{c}_1,\overline{c}_2))<\frac{\epsilon}{6}.$$

Step 5—Part b): Now, I bound the term (C.16). To do this, choose $\overline{e}$ to satisfy

$$\mathbb{P}(\chi^2_k>\overline{e})<\frac{\epsilon}{12}.$$

Since this selection of $\overline{e}$ implies that

$$\sup_{(\pi,F)\in\Pi^n_2(c^*)\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon}\backslash A\ \cap\ s's>\overline{e}\ \cap\ C^c(\underline{c}_1,\overline{c}_2))<\frac{\epsilon}{12},$$

it is sufficient to show that there is $\overline{c}_2$ large enough such that:

$$\sup_{(\pi,F)\in\Pi^n_2(c^*)\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon}\backslash A\ \cap\ s's\leq\overline{e}\ \cap\ C^c(\underline{c}_1,\overline{c}_2))<\frac{\epsilon}{12}$$

The key to establish this result is to show that the for t't large enough and $s's$ in a compact set, the test statistic $z(s,t)$ defined in Result 1 is close to the statistic $(s't)^2/(t't)$.

Step 5—Part c): Let $\mathbf{o}(t't)$ denote the function:

$$\mathbf{o}(t't)\equiv\Big(((s's/t't)-1)^2+4(s't)^2/(t't)^2\Big)^{1/2}-1.$$

Note that:

$$
\begin{aligned}
& 8\ln\Big[I_0\Big((t't/8)(1+\mathbf{o}(t't))\Big)\Big],\\
= \;& 8\ln\Big[\frac{e^{(t't/8)(1+\mathbf{o}(t't))}}{\sqrt{2pi((t't/8)(1+\mathbf{o}(t't)))}}\Big(1+O\Big(\frac{1}{(t't/8)(1+\mathbf{o}(t't))}\Big)\Big)\Big],\\
& \text{(where I have used the asymptotic approximation}\\
& \text{for } I_0(z) \text{ in p. 435 of Olver (1997) and the definition}\\
& \text{of} \sim \text{in p. 4 of the same book)}\\
= \;& 8\ln\Big[\frac{e^{(t't/8)(1+\mathbf{o}(t't))}}{\sqrt{2pi((t't/8)(1+\mathbf{o}(t't)))}}\Big]\\
+ \;& 8\ln\Big[\Big(1+O\Big(\frac{1}{(t't/8)(1+\mathbf{o}(t't))}\Big)\Big)\Big],\\
= \;& t't(1+\mathbf{o}(t't)) - 4\ln(2pi) - 4\ln(t't/8) - 4\ln(1+\mathbf{o}(t't))\\
+ \;& 8\ln\Big[\Big(1+O\Big(\frac{1}{(t't/8)(1+\mathbf{o}(t't))}\Big)\Big)\Big].
\end{aligned}
$$

Therefore, $z_{\text{WAP}}(s,t)$ in Result 1:

$$
(s's - t't) + 8\ln\Big[\,I_0\Big(\frac{1}{8}\Big[(s's-t't)^2 + 4(s't)^2)\Big]^{1/2}\Big)\,\Big] + 4\ln(2pi) + 4\ln((1/8)t't)
$$

can be written in terms of the Conditional Likelihood Ratio statistic (CLR) as follows:

$$
\begin{aligned}
z_{\text{WAP}}(s,t) \;\; \equiv \;\; & (s's - t't) + t't(1+\mathbf{o}(t't)) - 4\ln(1+\mathbf{o}(t't))\\
+ \;\; & 8\ln\Big[\Big(1+O\Big(\frac{1}{(t't/8)(1+\mathbf{o}(t't))}\Big)\Big)\Big],\\
= \;\; & 2\text{CLR}(s,t) - 4\ln(1+\mathbf{o}(t't))\\
+ \;\; & 8\ln\Big[\Big(1+O\Big(\frac{1}{(t't/8)(1+\mathbf{o}(t't))}\Big)\Big)\Big],\\
& \text{(where we have used the fact that } t't(1+\boldsymbol{o}(t't))\\
& \text{equals } [(s's-t't)^2 + (s't)]^{1/2}\ ).
\end{aligned}
$$

It is well-known for large values of $t't$ and for values of $s's$ in a compact set the CLR can be approximated by the LM statistic ($\equiv s't/t't$) uniformly over the values of s. Choose $\zeta^* > 0$ to satisfy:

$$
\mathbb{P}(-\eta^* \leq N(0,1) \leq \eta^*) = \frac{\epsilon}{24}.
$$

Therefore, using the same argument as in part b) of step 2 one can show for $\zeta^* > 0$ there is $\overline{c}_2^*$—that depends on ζ^*—such that uniformly over $s's \leq \overline{e}$

$$
|z_{\text{WAP}}(s,t) - c_{\text{WAP}}(t;\alpha) - 2LM(s,t) + 2\chi^2_{1,1-\alpha}| < \zeta^*,
$$

where $\chi^2_{1,1-\alpha}$ is the 1-α quantile of a chi-squared random variable with one degree of freedom and $c_{\text{WAP}}(t;\alpha)$ is the conditonal critical value of $z_{\text{WAP}}(s,t)$.

Step 5—Part d): Note that

$$x \equiv (s,t) \in (A^{\delta_\epsilon} \backslash A \, \cap \, s's \leq \overline{e} \, \cap \, t't > \overline{c}_2^*),$$

implies that:

$$z_(s,t) - c(t;\alpha) < 0,$$

and also that there is $x_0(\delta_\epsilon) \equiv (s_0, t_0) \in A$ such that $z(s_0,t_0) - c(t;\alpha) > 0$ and $d(x, x_0(\delta_\epsilon)) < \delta_\epsilon$. Since the test based on the test statistic $z(s,t)$ with conditional critical value $c(t;\alpha)$ is equivalent to the test based on $z_{\text{WAP}}(s,t)$ and $c_{\text{WAP}}(t;\alpha)$, it follows that:

$$z_{\text{WAP}}(s,t) - c_{\text{WAP}}(t;\alpha) < 0, \text{ and } z_{\text{WAP}}(s_0,t_0) - c_{\text{WAP}}(t;\alpha) > 0.$$

Consequently:

$$\text{LM}(s,t) - \chi^2_{1,1-\alpha} < \zeta^*/2$$

and

$$\text{LM}(s_0,t_0) - \chi^2_{1,1-\alpha} > -\zeta^*/2.$$

Note that the LM statistic can be written as a function of (S, ω_t) where $\omega_t \equiv t/||t||$. Since the partial derivatives of $(s'\omega_t)^2$ are bounded whenever $s's \leq \overline{e}$, the LM statistic satisfies the Lipschitz condition when s's belongs to the desired domain. Let $M(\overline{e})$ denote the Lipschitz constant of the LM statistic. Since:

$$-d(x, x_0(\delta_\epsilon))M(\overline{e}) \leq \text{LM}(s,t) - \text{LM}(s_0,t_0) \leq M(\overline{e})d(x, x_0(\delta_\epsilon)),$$

then:

$$\begin{aligned} -\delta_\epsilon M(\overline{e}) &\leq -d(x, x_0(\delta_\epsilon))M(\overline{e}) \\ &\leq \text{LM}(s,t) - \chi^2_{1,1-\alpha} + \chi^2_{1,1-\alpha} - \text{LM}(s_0,t_0), \\ &\leq \text{LM}(s,t) - \chi^2_{1,1-\alpha} + \zeta^*/2, \\ &\quad \text{(where I have used the fact that } \text{LM}(s_0,t_0) - \chi^2_{1,1-\alpha} > -\zeta^*/2), \\ &\leq \zeta^*, \\ &\quad \text{(where I have used the fact that } \text{LM}(s,t) - \chi^2_{1,1-\alpha} < \zeta^*/2). \end{aligned}$$

One can further shrink δ_ϵ to satisfy $\delta_\epsilon M(\overline{e}) < -\zeta^*$. This means that:

$$\begin{aligned} \sup_{(\pi,F)\in\Pi_2^n(c^*)\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon} \backslash A \, \cap \, s's \leq \overline{e} \, \cap \, t't > \overline{c}_2^*) &\leq \sup_{(\pi,F)\in\Pi_2^n(c^*)\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(-\eta^* \leq \text{LM}(s,t) \leq \eta^*) \\ &= \mathbb{P}(-\eta^* \leq N(0,1) \leq \eta^*) \\ &= \frac{\epsilon}{24} \end{aligned}$$

Since:

$$\sup_{(\pi,F)\in\Pi_2^n(c^*)\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon} \backslash A \, \cap \, s's \leq \overline{e} \, \cap \, C^c(\underline{c}_1, \overline{c}_2^*)),$$

is smaller than or equal:

$$\sup_{(\pi,F)\in\Pi_2^n(c^*)\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon} \backslash A \, \cap \, s's \leq \overline{e} \, \cap \, t't \leq \underline{c}_1),$$

plus

$$\sup_{(\pi,F)\in\Pi_2^n(c^*)\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon} \backslash A \, \cap \, s's \leq \overline{e} \, \cap \, t't > \underline{c}_2^*),$$

it follows that:

$$\sup_{(\pi,F)\in\Pi_2^n(c^*)\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon}\backslash A \,\cap\, s's\leq \overline{e} \,\cap\, C^c(\underline{c}_1,\overline{c}_2^*)) < \frac{\epsilon}{24}+\frac{\epsilon}{24}=\frac{\epsilon}{12}.$$

This implies that:

$$\sup_{(\pi,F)\in\Pi_2^n(c^*)\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon}\backslash A\, \cap C^c(\underline{c}_1,\overline{c}_2)) < \frac{\epsilon}{12}+\frac{\epsilon}{12}=\frac{\epsilon}{6}.$$

<u>Main conclusion of Step 5</u>: The conclusion of Step 5 is that there are constants $\underline{c}_1^*,\overline{c}_2^*$ such that:

$$\sup_{(\pi,F)\in\mathbb{R}^k\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon}\backslash A)\leq\frac{2\epsilon}{3}+\sup_{(\pi,F)\in\mathbb{R}^k\times\mathcal{F}} Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon}\backslash A\, \cap C^c(\underline{c}_1^*,\overline{c}_2^*))\leq\frac{2\epsilon}{3}+\frac{\epsilon}{3}=\epsilon.$$

Step 1 to Step 5: I have shown that for every $\epsilon>0$ one can choose constants $\underline{b}_1^*,\overline{b}_2^*$, $\underline{c}_1^*,\overline{c}_2^*$, $\underline{d}_1^*$, $\overline{d}_2^*$ such that for any sample size and $(\pi,F)\in\mathbb{R}^k\times\mathcal{F}$:

$$\begin{aligned}
Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta_\epsilon}\backslash A) &\leq Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta}\backslash A\, \cap[B(\underline{b}_1^*,\overline{b}_1^*)\cap C(\underline{c}_1^*,\overline{c}_2^*)\cap D(\underline{d}_1^*,\overline{d}_2^*)])\\
&+\mathbb{P}(\chi^2_k\notin(\underline{b}_1^*,\overline{b}_2^*))+\mathbb{P}(\chi^2_1\notin(\overline{d}_1^*,\overline{d}_2^*))\\
&+Q^n_{\beta_0,\pi\Sigma(F)}(A^{\delta}\backslash A\, \cap C^c(\underline{c}_1^*,\overline{c}_2^*))\\
&\leq \epsilon
\end{aligned}$$

Since for any $\delta>0$:

$$\begin{aligned}
P^n_{\beta_0,\pi,\mathcal{F}}(A) &\leq Q^n_{\beta_0,\pi,\Sigma(F)}(A)+\frac{1}{\delta}d_{\mathrm{BL}_1}(P^n_{\beta_0,\pi,F},Q^n_{\beta_0,\pi,\Sigma(F)})+Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta}\backslash A),\\
&= \alpha+\frac{1}{\delta}d_{\mathrm{BL}_1}(P^n_{\beta_0,\pi,F},Q^n_{\beta_0,\pi,\Sigma(F)})+Q^n_{\beta_0,\pi,\Sigma(F)}(A^{\delta}\backslash A),
\end{aligned}$$

and, by assumption, there is $M_\epsilon\in\mathbb{N}$ such that for any $n\geq M_\epsilon$:

$$d_{\mathrm{BL}_1}(P^n_{\beta_0,\pi,F},Q^n_{\beta_0,\pi,\Sigma(F)})\leq\epsilon\delta_\epsilon,$$

I conclude that for $n\geq M_\epsilon$:

$$\sup_{(\pi,F)\in\mathbb{R}^k\times\mathcal{F}} P^n_{\beta_0,\pi,\mathcal{F}}(A)\leq\alpha+2\epsilon.$$

This establishes the asymptotic validity of the test in Result 1.

Final Comments: The asymptotic validity of the test in Result 1 exploited specific properties of the test statistic $z_{\text{WAP}}(s,t)$. For example, I showed that $z_{\text{WAP}}(s,t)$ is a Lipschitz function function of (s,t) provided $s's$, $t't$, and $(s't)^2/t't$ are bounded from below and from above. Under the same domain restriction, the proof also uses the existence of a threshold $\gamma_\epsilon>0$ such that the probability of the event $-\gamma_\epsilon\leq z_{\text{WAP}}(S,T)-c_{\text{WAP}}(T,\alpha)\leq 0$ is smaller than ϵ, uniformly over the family of distributions $Q^n_{\beta_0,\pi,F}$. Finally, the proof used an asymptotic approximation of the modified Bessel function of the first kind for large values of its argument that appears in Olver (1997), p. 435. Under this approximation, the $z_{\text{WAP}}(s,t)$ is approximately $2CLR(s,t)$ when $t't$ is large.

C.2. *Local Asymptotic Power of the test in Result 1*

Now I derive the local asymptotic power of the test in Result 1 under the following assumption:

Assumption L1: The class of distributions $\mathcal{F}$ is such that:

$$\lim_{n\to\infty} d_{\mathrm{BL}}\Big(P^n_{\beta_0+\frac{c}{\sqrt{n}},\pi,F}, Q^n_{\beta_0+\frac{c}{\sqrt{n}}}\Big) \to 0$$

This is, the Bounded Lipschitz distance between the measures $P^n_{\beta,\pi,F}$ and $Q^n_{\beta,\pi,\Sigma(F)}$ coverges to zero as the sample size grows large for any local alternative of the form $\beta_0 + c/\sqrt{n}$.

Asymptotic Efficiency of the test in Result 1: Suppose that Assumption L1 holds and suppose that there are constants $0 < \underline{\lambda} < \overline{\lambda} < \infty$ such that the eigenvalues of $\Sigma(F)$ belong to an interval $[\underline{\lambda}, \overline{\lambda}]$ for any $F \in \mathcal{F}$. If $\Sigma(F) = \Psi(F) \otimes \Phi(F)$ and $\pi \neq \mathbf{0}_{k\times 1}$, then:

$$\liminf_{n\to\infty} P^n_{\beta_0+\frac{c}{\sqrt{n}},\pi,F}(z_{\mathrm{WAP}}(s,t) - c_{\mathrm{WAP}}(t;\alpha) \geq 0) \geq \mathbb{P}\Big(\chi^2_1(\mu^2(\beta_0, c, \pi, F)) > \chi^2_{1,1-\alpha}\Big),$$

where $\chi^2_1(\mu^2(\beta_0, c, \pi, F))$ is a non-central chi-square distribution with centrality parameter:

$$\mu^2(\beta_0, c, \pi, F) \equiv c^2(\pi'\Phi(F)^{-1}\pi)(b_0'\Psi(F)b_0)^{-1}.$$

I establish the Local efficiency of the test in Result 1 in 6 steps:

Step 0: Fix $(\pi, F) \in \mathbb{R}^k \times \mathcal{F}$ and suppose that $||\pi|| \neq 0$. Let $\overline{e}$ denote a positive scalar. In Part c) of Step 5 I have shown that for any $\zeta > 0$ there is $\underline{e}(\eta)$ such that for any $t't > \underline{e}(\eta)$—and uniformly over the values of $s's < \overline{e}$—it follows that:

$$|z_{\mathrm{WAP}}(s,t) - c_{\mathrm{WAP}}(t;\alpha) - 2\mathrm{LM}(s,t) + 2\chi^2_{1,1-\alpha}| < \zeta/2,$$

where $\mathrm{LM}(s,t) \equiv (s't)^2/t'^t$. This means that if $s's < \overline{e}$ and $t't > \underline{e}(\zeta)$:

$$LM(s,t) - \chi^2_{1,1-\alpha} > \zeta/4 \implies z_{\mathrm{WAP}}(s,t) - c_{\mathrm{WAP}}(t;\alpha) \geq 0.$$

Therefore, for any local alternative $\beta(c) \equiv \beta_0 + \frac{c}{\sqrt{n}}$:

$$\begin{aligned}
P^n_{\beta(c),\pi,F}(z_{\mathrm{WAP}}(s,t) - c_{\mathrm{WAP}}(t;\alpha) \geq 0) \;&\geq\; P^n_{\beta(c),\pi,F}(LM(s,t) \geq \chi^2_{1-\alpha} + \zeta \text{ and } s's < \overline{e} \text{ and } t't > \underline{e}(\zeta)),\\
&\geq\; P^n_{\beta(c),\pi,F}(LM(s,t) \geq \chi^2_{1-\alpha} + \zeta/4)\\
&+\; P^n_{\beta(c),\pi,F}(s's < \overline{e})\\
&+\; P^n_{\beta(c),\pi,F}(t't > \underline{e}(\zeta)) - 2,\\
&\;\text{(where I have used } P(A\cap B) = P(A) + P(B) - P(A\cup B)\\
&\;\text{twice, and also the fact that } P(A\cup B) \leq 1).
\end{aligned}$$

I now characterize the asymptotic behavior of the terms:

$$P^n_{\beta(c),\pi,F}(LM(s,t) \geq \chi^2_{1-\alpha} + \zeta/4), \quad P^n_{\beta(c),\pi,F}(s's < \overline{e}), \quad P^n_{\beta(c),\pi,F}(t't > \underline{e}(\zeta)).$$

Step 1: Consider first the term:

$$P^n_{\beta(c),\pi,F}(LM(s,t) \geq \chi^2_{1-\alpha} + \zeta/4).$$

Note that the event:

$$E_1 = \{(s', t')' \in \mathbb{R}^{2k} \mid LM(s,t) > \chi^2_{1,1-\alpha} + \zeta/4\},$$

is the same as the event:

$$E_1 = \{(s', t')' \in \mathbb{R}^{2k} \mid LM(s, t/\sqrt{n}) > \chi^2_{1,1-\alpha} + \zeta/4\},$$

as $\text{LM}(s,t) = (s't)^2/t't = s't/\sqrt{n})^2/(t/\sqrt{n})'(t/\sqrt{n})$.

Let $\widetilde{P}^n_{\beta(c),\pi,F}$ denote the distribution of the random vector $(S', (t/\sqrt{n})')'$. Since the transformation $(x,y) \to (x, y/\sqrt{n})$ is Lipschitz for any $n \in \mathbb{N}$ (with constant 1) it follows—by assumption—that as $n \to \infty$:[15]

$$d_{\text{BL}_1}(\widetilde{P}^n_{\beta(c),\pi,F}, \widetilde{Q}^n_{\beta(c),\pi,\Sigma(F)}) \to 0,$$

where $\widetilde{Q}$ is the distribution of:

$$\widetilde{Q}^n_{\beta,\pi,\Sigma(F)} \equiv \mathcal{N}_{2k}\left(\begin{matrix} ([b_0' \otimes \mathbb{I}_k]\Sigma(b_0 \otimes \mathbb{I}_k)]^{-1/2} c\pi \\ [(a_0' \otimes \mathbb{I}_k)\Sigma(F)^{-1}(a_0 \otimes \mathbb{I}_k)]^{-1/2}(a_0' \otimes \mathbb{I}_k)\Sigma(F)^{-1}(a_n(c) \otimes \mathbb{I}_k)\pi \end{matrix}, \begin{pmatrix} \mathbb{I}_k & \mathbf{0}_{k\times k} \\ \mathbf{0}_{k\times k} & \mathbb{I}_k/\sqrt{n} \end{pmatrix} \right),$$

with $a_n(c) = (\beta_0 + c/\sqrt{n}, 1)'$. Moreover, the convergence of the mean vector of this distribution and its covariance matrix this implies that:

$$d_{\text{BL}_1}(\widetilde{Q}^n_{\beta(c),\pi,\Sigma(F)}, Q_{\beta_0,\pi,\Sigma(F)}) \to 0$$

where

$$Q_{\beta_0,\pi,\Sigma(F)} \equiv \mathcal{N}_{2k}\left(\begin{matrix} ([b_0' \otimes \mathbb{I}_k]\Sigma(b_0 \otimes \mathbb{I}_k)]^{-1/2} c\pi \\ [(a_0' \otimes \mathbb{I}_k)\Sigma(F)^{-1}(a_0 \otimes \mathbb{I}_k)]^{1/2}\pi \end{matrix}, \begin{pmatrix} \mathbb{I}_k & \mathbf{0}_{k\times k} \\ \mathbf{0}_{k\times k} & \mathbf{0}_{k\times k} \end{pmatrix} \right).$$

Thus, one can conclude that

$$d_{\text{BL}_1}(\widetilde{P}^n_{\beta(c),\pi,\Sigma(F)}, Q_{\beta_0,\pi,\Sigma(F)}) \to 0.$$

For any event, and in particular for E_1,

$$\widetilde{P}^n_{\beta(c),\pi,F}(E_1) \geq Q_{\beta_0,\pi,\Sigma(F)}(E_1) - \frac{1}{\delta} d_{\text{BL}_1}(\widetilde{P}^n_{\beta(c),\pi,F}, Q_{\beta_0,\pi,\Sigma(F)}) - Q_{\beta_0,\pi,\Sigma(F)}((E_1^c)^\delta \backslash E_1^c)$$

Under the probability measure $Q_{\beta_0,\pi,\Sigma(F)}$ (which does not depend on n), the topological boundary of E_1^c

$$\text{Bd}(E_1^c) \equiv \{(s',t')' \in \mathbb{R}^{2k} \mid LM(s,t) = \chi^2_{1,1-\alpha} + \zeta/4\},$$

has probability zero. Therefore, there exists $\delta_{\epsilon,\zeta}$ (independent of n) such that:

$$Q_{\beta_0,\pi,F}((E^c)_!^{\delta_{\epsilon,\zeta}} \backslash E_1^c) < \frac{\epsilon}{6}.$$

Moreover, by choosing $N(\epsilon,\zeta) \in \mathbb{N}$ to be such that for $n \geq N(\epsilon,\zeta)$:

$$d_{\text{BL}_1}(\widetilde{P}^n_{\beta(c),\pi,\Sigma(F)}, Q_{\beta_0,\pi,\Sigma(F)}) \leq \delta_{\epsilon,\zeta}\frac{\epsilon}{6},$$

[15] To see this, note that for any function $h \in \text{BL}_1$ we have that:

$$\left| \mathbb{E}_{\widetilde{P}^n_{\beta(c),\pi,F}}[h(X)] - \mathbb{E}_{\widetilde{Q}^n_{\beta(c),\pi,F}}[h(X)] \right| = \left| \mathbb{E}_{P^n_{\beta(c),\pi,F}}[h \circ g_n(S,T)] - \mathbb{E}_{Q^n_{\beta(c),\pi,F}}[h \circ g_n(S,T)] \right|,$$

where $g_n(s,t) = (s', (t/\sqrt{n})')'$ is an element of BL_1. Consequently,

$$0 \leq d_{\text{BL}_1}(\widetilde{P}^n_{\beta(c),\pi,F}, \widetilde{Q}^n_{\beta(c),\pi,\Sigma(F)}) \leq d_{\text{BL}_1}(P^n_{\beta(c),\pi,F}, Q^n_{\beta(c),\pi,\Sigma(F)}).$$

we have that:

$$\begin{aligned} P^n_{\beta(c),\pi,F}(E_1) &= \widetilde{P}^n_{\beta(c),\pi,F}(E_1), \\ &\geq Q_{\beta_0,\pi,\Sigma(F)}(E_1) - \frac{1}{\delta_{\epsilon,\zeta}} d_{\mathrm{BL}_1}(\widetilde{P}^n_{\beta(c),\pi,F}, Q_{\beta_0,\pi,\Sigma(F)}) - Q_{\beta_0,\pi,\Sigma(F)}((E_1^c)^{\delta_{\epsilon,\zeta}} \backslash E_1^c), \\ &\geq Q_{\beta_0,\pi,\Sigma(F)}(E_1) - \frac{\epsilon}{6} - \frac{\epsilon}{6}, \\ &= Q_{\beta_0,\pi,\Sigma(F)}(E_1) - \frac{\epsilon}{3}. \end{aligned}$$

Moreover, under the probability measure $Q_{\beta_0,\pi,\Sigma(F)}(E_1)$:

$$\frac{t'S}{\sqrt{t't}} \sim \mathcal{N}\left(\mu(\beta_0, c, \pi, F)\ ,\ 1\right),$$

where

$$\mu(\beta_0, c, \pi, F) \equiv c \frac{\pi'\Big([(a_0' \otimes \mathbb{I}_k)]\Sigma(F)^{-1}(a_0 \otimes \mathbb{I}_k)]^{1/2}[(b_0' \otimes \mathbb{I}_k)\Sigma(F)(b_0 \otimes \mathbb{I}_k)]^{-1/2}\Big)\pi}{\Big(\pi'[(a_0' \otimes \mathbb{I}_k)]\Sigma^{-1}(F)(a_0 \otimes \mathbb{I}_k)]\pi\Big)^{1/2}}.$$

This means that $Q_{\beta_0,\pi,\Sigma(F)}(E_1)$ is the same as:

$$\mathbb{P}\Big(\chi^2_1(\mu^2(\beta_0, c, \pi, F)) > \chi^2_{1,1-\alpha} + \eta\Big).$$

This means that for any $\zeta > 0, \epsilon > 0$ there is $N(\epsilon, \zeta) \in \mathbb{N}$ such that for $n \geq N(\epsilon, \zeta)$:

$$P^n_{\beta(c),\pi,F}(E_1) \geq \mathbb{P}\Big(\chi^2_1(\mu^2(\beta_0, c, \pi, F)) > \chi^2_{1,1-\alpha} + \zeta\Big) - \frac{\epsilon}{3},$$

where $\chi^2_1(\mu^2(c, \beta_0, c, \pi, F))$ is a non-central chi-square distribution with centrality parameter:

$$\mu(\beta_0, c, \pi, F) \equiv c \frac{\pi'\Big([(a_0' \otimes \mathbb{I}_k)]\Sigma(F)^{-1}(a_0 \otimes \mathbb{I}_k)]^{1/2}[(b_0' \otimes \mathbb{I}_k)\Sigma(F)(b_0 \otimes \mathbb{I}_k)]^{-1/2}\Big)\pi}{\Big(\pi'[(a_0' \otimes \mathbb{I}_k)]\Sigma^{-1}(F)(a_0 \otimes \mathbb{I}_k)]\pi\Big)^{1/2}}.$$

Step 2: Now I take care of the term $P^n_{\beta(c),\pi,F}(s's < \overline{e})$. In particular, I show that $\overline{\epsilon}$ to make this term to be at least $1 - \epsilon/3$ for n large enough. Let E_2 denote the event:

$$E_2 \equiv \{(s', t')' \mid s's < \overline{e}\}$$

Note that for any $\delta > 0$:

$$\begin{aligned} P^n_{\beta(c),\pi,F}(E_2) &\geq Q^n_{\beta(c),\pi,\Sigma(F)}(E_2) - \frac{1}{\delta} d_{\mathrm{BL}_1}(P^n_{\beta(c),\pi,F}, Q^n_{\beta(c),\pi,\Sigma(F)}) - Q^n_{\beta(c),\pi,\Sigma(F)}((E_2^c)^{\delta} \backslash E^c) \\ &= \mathbb{P}_S(E_2) - \frac{1}{\delta} d_{\mathrm{BL}_1}(P^n_{\beta(c),\pi,F}, Q^n_{\beta(c),\pi,\Sigma(F)}) - \mathbb{P}_S((E_2^c)^{\delta} \backslash E_2^c), \end{aligned}$$

where $S \sim \mathcal{N}_k([b_0' \otimes \mathbb{I}_k \Sigma(F)(b_0 \otimes \mathbb{I}_k]^{-1/2}\pi c\ ,\ \mathbb{I}_{2k})$. Once again, since the topological boundary of E_2 has zero measure under $\mathbb{P}_S$ there exists $\delta_{\epsilon,\overline{e}}$ such that:

$$\mathbb{P}_S((E_2^c)^{\delta_{\epsilon,\overline{e}}} \backslash E_2^c) < \frac{\epsilon}{9}.$$

This means that one can choose $N(\epsilon, \overline{e}) \in \mathbb{N}$ such that for $n \geq N(\epsilon, \overline{e})$:

$$P^n_{\beta(c),\pi,F}(E_2) \geq \mathbb{P}_S(E_2) - \frac{2\epsilon}{9}.$$

Moreover, there is $\overline{e}^*$ large enough such that $\mathbb{P}_S(E_2) \geq 1 - \frac{\epsilon}{9}$. Therefore, for $n \geq N(\epsilon, \overline{e}^*)$

$$P^n_{\beta(c),\pi,F}(E_2) \geq 1 - \frac{\epsilon}{3}.$$

Step 3: Finally, consider the term $P^n_{\beta(c),\pi,F}(t't > \underline{e})$. I show that for any fixed $\overline{\epsilon}$ there is n large enough such that this term is at least $1 - \epsilon/3$. Define:

$$E_{3,n} \equiv \{(s', t')' \mid t't > \underline{e}/\sqrt{n}\}.$$

Note that:

$$\begin{aligned}
P^n_{\beta(c),\pi,F}(t't > \underline{e}) &= \widetilde{P}^n_{\beta(c),\pi,F}(E_{3,n}), \\
&= Q_{\beta_0,\pi,\Sigma(F)}(E_{3,n}) - \frac{1}{\delta} d_{\mathrm{BL}_1}(\widetilde{P}^n_{\beta(c),\pi,F}, Q_{\beta_0,\pi,\Sigma(F)}) - Q_{\beta_0,\pi,\Sigma(F)}((E^c_{3,n})^\delta \backslash E^c_{3,n}). \\
&\geq Q_{\beta_0,\pi,\Sigma(F)}(E_{3,n}) - \frac{1}{\delta} d_{\mathrm{BL}_1}(\widetilde{P}^n_{\beta(c),\pi,F}, Q_{\beta_0,\pi,\Sigma(F)}) - Q_{\beta_0,\pi,\Sigma(F)}((E^c_{3,n})^\delta).
\end{aligned}$$

Under $Q_{\beta_0,\pi,\Sigma(F)}$, the statistic t has a degenerate distribution with all of its mass at $t^* \equiv [(a_0' \otimes \mathbb{I}_k)]\Sigma^{-1}(F)(a_0 \otimes \mathbb{I}_k)]^{1/2}\pi \equiv 0$. This means that for n large enough:

$$Q_{\beta_0,\pi,\Sigma(F)}(E_{3,n}) = \mathbf{1}\{t^* \in E_{3,n}\} = 1.$$

Take any δ such the ball of radius δ^* around t^*, $B_{\delta^*}(t^*)$, that excludes the origin. Note that for any n such that $e/\sqrt{n} < \delta^*$:

$$B_{\underline{e}^{1/2}/n^{1/4}}(\mathbf{0}_{k\times 1}) \cap B_{\delta^*}(t^*) = \emptyset.$$

This means that for n large enough there is no $t_0 \in \mathbb{R}^k$ such that: $t_0't_0 < \underline{e}/\sqrt{n}$ and $d(t_0, t^*) \leq \delta^*$. This means that for n large enough $t^* \notin (E^c_{3,n})^\delta$. Therefore for n large enough:

$$\begin{aligned}
P^n_{\beta(c),\pi,F}(t't > \underline{e}) &\geq 1 - \frac{1}{\delta^*} d_{\mathrm{BL}_1}(\widetilde{P}^n_{\beta(c),\pi,F}, Q_{\beta_0,\pi,\Sigma(F)}) - 0 \\
&\geq 1 - \frac{\epsilon}{3}.
\end{aligned}$$

Step 4: I have shown that:

$$\begin{aligned}
P^n_{\beta(c),\pi,F}(z(s,t) - c(t;\alpha) \geq 0) &\geq P^n_{\beta(c),\pi,F}(LM(s,t) \geq \chi^2_{1-\alpha} + \zeta/4) \\
&+ P^n_{\beta(c),\pi,F}(s's < \overline{e}^*) \\
&+ P^n_{\beta(c),\pi,F}(t't > \underline{e}(\zeta)) - 2.
\end{aligned}$$

Step 1, 2, 3 of this proof have established the existence of $N(\epsilon, \zeta, \overline{e}^*)$ such that for any $n \geq N(\epsilon, \zeta, \overline{e}^*)$:

$$\begin{aligned}
P^n_{\beta(c),\pi,F}(LM(s,t) \geq \chi^2_{1-\alpha} + \zeta/4) &\geq \mathbb{P}\Big(\chi^2_1(\mu^2(\beta_0, c, \pi, F)) > \chi^2_{1,1-\alpha} + \zeta\Big) - \frac{\epsilon}{3}, \\
P^n_{\beta(c),\pi,F}(s's < \overline{e}^*) &\geq 1 - \frac{\epsilon}{3}, \\
P^n_{\beta(c),\pi,F}(t't > \underline{e}(\zeta)) &\geq 1 - \frac{\epsilon}{3},
\end{aligned}$$

where the centrality parameter $\mu^2(\beta_0, c, \pi, F)$ is given by:

$$\mu^2(\beta_0, c, \pi, F) \equiv \left(c \frac{\pi'\Big([(a_0' \otimes \mathbb{I}_k)]\Sigma(F)^{-1}(a_0 \otimes \mathbb{I}_k)]^{1/2}[(b_0' \otimes \mathbb{I}_k)\Sigma(F)(b_0 \otimes \mathbb{I}_k)]^{-1/2}\Big)\pi}{\Big(\pi'[(a_0' \otimes \mathbb{I}_k)]\Sigma^{-1}(F)(a_0 \otimes \mathbb{I}_k)]\pi\Big)^{1/2}} \right)^2.$$

This implies that:

$$\liminf_{n\to\infty} P^n_{\beta_0+\frac{c}{\sqrt{n}},\pi,F}(z_{\text{WAP}}(s,t) - c_{\text{WAP}}(t;\alpha) \geq 0) \geq \mathbb{P}\Big(\chi^2_1(\mu^2(\beta_0,c,\pi,F)) > \chi^2_{1,1-\alpha} + \zeta\Big),$$

for any $\zeta > 0$. Since the distribution on the right-hand side is continuous in ζ, then:

$$\liminf_{n\to\infty} P^n_{\beta_0+\frac{c}{\sqrt{n}},\pi,F}(z_{\text{WAP}}(s,t) - c_{\text{WAP}}(t;\alpha) \geq 0) \geq \mathbb{P}\Big(\chi^2_1(\mu^2(\beta_0,c,\pi,F)) > \chi^2_{1,1-\alpha}\Big)$$

Step 5: If $\Sigma(F) = \Psi(F) \otimes \Phi(F)$:

$$[(a_0' \otimes \mathbb{I}_k)]\Sigma(F)^{-1}(a_0 \otimes \mathbb{I}_k)] = (a_0'\Psi^{-1}a_0)\Phi^{-1}$$

and

$$[(b_0' \otimes \mathbb{I}_k)\Sigma(F)(b_0 \otimes \mathbb{I}_k)] = (b_0'\Psi b_0)\Phi.$$

Therefore, the expression for the centrality parameter simplifies to:

$$\mu^2(\beta_0,c,\pi,F) \equiv c^2(\pi'\Phi^{-1}\pi)(b_0'\Psi b_0)^{-1}.$$

Final Comments: The lower bound on local power above implies that the test in Result 1 is as powerful (locally) as a GMM-Wald test for β_0 based on the sample moment condition:

$$\frac{1}{\sqrt{n}}Z'(y - \beta_0 x) = \mathbf{0}.$$

To see this, note that under Assumption L0 the asymptotic variance of the sample moment condition is simply QW_0Q where Q is the probability limit of $Z'Z/n$ and $W_0 \equiv (b_0' \otimes \mathbb{I}_k)\Sigma(b_0 \otimes \mathbb{I}_k)$. Therefore, the efficient GMM estimator for β (assuming W_0 is known) is:

$$\begin{aligned}\beta_{\text{GMM}} &= \Big(X'Z(Z'ZW_0Z'Z)^{-1}Z'X\Big)^{-1}X'Z(Z'ZW_0Z'Z)^{-1}Z'y \\ &= \Big(\widehat{\gamma}_2 W_0^{-1}\widehat{\gamma}_2\Big)^{-1}\widehat{\gamma}_2 W_0^{-1}\widehat{\gamma}_1.\end{aligned}$$

The efficient α-level GMM-Wald test for $\beta = \beta_0$ rejects whenever:

$$\Big(\Big(\widehat{\gamma}_2 W_0^{-1}\widehat{\gamma}_2\Big)^{-1/2}\widehat{\gamma}_2 W_0^{-1}\sqrt{n}(\widehat{\gamma}_1 - \beta_0\widehat{\gamma}_2)\Big)^2 > \chi^2_{1,1-\alpha},$$

and this test has local power, under alternatives of the form $\beta_0 + c/\sqrt{n}$, given by:

$$\mathbb{P}\Big(\chi^2_1(c^2(\pi' W_0^{-1}\pi)) > \chi^2_{1,1-\alpha}\Big).$$

If Σ is of the form $\Psi \otimes \Omega$, then $c^2(\pi' W_0^{-1}\pi)$ coincides with the centrality parameter $\mu^2(\beta_0,c,\pi,F)$.