Admissible, Similar Tests: A characterization. Appendix B. Proof of Result 1.

APPENDIX B: KRONECKER CASE

B.1. *Weights for (β,π) in the Kronecker case*

This section analyzes the properties of the weights:

$$\begin{pmatrix}\beta\pi\\ \pi\end{pmatrix} = n^{-1/2}\left(\Psi_\Sigma \widetilde{C}_0' \otimes \Phi_\Sigma^{1/2}\right)\rho(\phi\otimes\omega),\ \widetilde{C}_0 \equiv \begin{pmatrix}(b_0'\Psi_\Sigma b_0)^{-1/2}b_0'\\ (a_0'\Psi_\Sigma^{-1}a_0)^{-1/2}a_0'\Psi_\Sigma^{-1}\end{pmatrix}, \tag{B.1}$$

with

$$\phi\sim\mathcal{U}(\mathcal{S}^1),\quad \omega\sim\mathcal{U}(\mathcal{S}^{k-1}), \tag{B.2}$$

and

$$\rho|\phi,\omega \sim \sqrt{\chi^2_k}\,/\,(\phi'\otimes\omega')\left(\widetilde{C}_0\Psi_\Sigma'\otimes{\Phi_\Sigma'}^{1/2}\right)\Sigma^{-1}\left(\Psi_\Sigma\widetilde{C}_0'\otimes\Phi_\Sigma^{1/2}\right)(\phi\otimes\omega). \tag{B.3}$$

The main assumption of this section is that $\Sigma=\Psi\otimes\Phi$. Note first that when $\Sigma=\Psi\otimes\Phi$:

$$\Psi_\Sigma = (\mathrm{vec}(\Phi)'\mathrm{vec}(\Phi))^{1/2}\Psi,\quad \Phi_\Sigma = \Phi/(\mathrm{vec}(\Phi)'\mathrm{vec}(\Phi))^{1/2}.$$

Therefore, we can write the weights in (B.1) as:

$$\begin{pmatrix}\beta\pi\\ \pi\end{pmatrix} = n^{-1/2}\left(\Psi C_0'\otimes\Phi\right)\rho(\phi\otimes\omega),\ C_0\equiv\begin{pmatrix}(b_0'\Psi b_0)^{-1/2}b_0'\\ (a_0'\Psi^{-1}a_0)^{-1/2}a_0'\Psi^{-1}\end{pmatrix}, \tag{B.4}$$

Weight for β: Under (B.4), the parameter β equals:

$$\beta \quad = \quad \frac{[1,0]\Psi C_0'\phi}{[0,1]\Psi C_0'\phi},$$

This ratio can be simplified using the following equalities. First:

$$\begin{aligned}
[1,0]\Psi C_0'\phi \quad &= \quad [1,0]\Psi\left[b_0(b_0'\Psi b_0)^{-1/2}\ ,\ \Psi^{-1}a_0(a_0'\Psi^{-1}a_0)^{-1/2}\right]\phi,\\
&\quad\quad \text{(by definition of } C_0)\\
&= \quad \left[[1,0]\Psi b_0(b_0'\Psi b_0)^{-1/2}\ ,\ \beta_0(a_0'\Psi^{-1}a_0)^{-1/2}\right]\phi,\\
&\quad\quad \text{(as } [1,0]a_0=\beta_0).
\end{aligned}$$

Second:

$$a_0'\Psi^{-1}a_0 = \frac{1}{\det(\Psi)}a_0'\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\Psi\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}a_0 = \det\left(\begin{pmatrix}1 & -\beta_0\\ 0 & 1\end{pmatrix}\Psi\begin{pmatrix}1 & -\beta_0\\ 0 & 1\end{pmatrix}'\right)^{-1} b_0'\Psi b_0,$$

and

$$1-r(\beta_0)^2 = \det\left(\begin{pmatrix}1 & -\beta_0\\ 0 & 1\end{pmatrix}\Psi\begin{pmatrix}1 & -\beta_0\\ 0 & 1\end{pmatrix}'\right)\Big/(b_0'\Psi b_0)([0,1]\Psi[0,1]'),$$

where $r(\beta_0)$ refers to the correlation coefficient of $(b_0';[0,1])\Psi(b_0';[0,1])'$. This means that the numerator for β is given by:

$$\begin{aligned}
[1,0]\Psi C_0'\phi \quad &= \quad [1,0]\Psi b_0(b_0'\Psi b_0)^{-1/2}\phi_1 + \beta_0\sqrt{1-r^2(\beta_0)}([0,1]\Psi[0,1]')^{1/2}\phi_2,\\
&= \quad [1,0]\Psi b_0(b_0'\Psi b_0)^{-1/2}\phi_1 - \beta_0([0,1]\Psi b_0)(b_0'\Psi b_0)^{-1/2}\phi_1\\
&+ \quad \beta_0\left(r(\beta_0)\phi_1+\sqrt{1-r^2(\beta_0)}([0,1]\phi_2\right)([0,1]\Psi[0,1]')^{1/2},\\
&\quad\quad \text{(where I have added and subtracted } \beta_0 r(\beta_0)([0,1]\Psi[0,1]')^{1/2}\phi_1).
\end{aligned}$$

Therefore:

$$[1,0]\Psi C_0'\phi = (b_0'\Psi b_0)^{1/2}\phi_1 + \beta_0[0,1]\Psi C_0'\phi,$$

where I have used the fact that:

$$[0,1]\Psi C_0'\phi = \left(r(\beta_0)\phi_1 + \sqrt{1-r^2(\beta_0)}([0,1]\phi_2\right)([0,1]\Psi[0,1]')^{1/2}.$$

This means that β can be written as:

$$\beta = \frac{[1,0]\Psi C_0'\phi}{[0,1]\Psi C_0'\phi} = \frac{(b_0'\Psi b_0)^{1/2}\phi_1}{[0,1]\Psi C_0'\phi} + \beta_0. \tag{B.5}$$

Weight for π: The distribution of the first-stage coefficient is given by:

$$\sqrt{n}\pi = ([0,1]\Psi C_0'\phi)\Phi^{1/2}\rho\omega, \tag{B.6}$$

This means that:

$$\sqrt{n}\pi \mid \phi \sim \mathcal{N}_k\left(\mathbf{0}\,,\ ([0,1]\Psi C_0'\phi)^2\Phi\right).$$

Comparison to the MM2 weights: I claim that if $\Sigma = \Psi \otimes \Phi$ the weights in (3.3) and (3.4) are equivalent to the 'MM2' weights proposed in MM15. To see this, note that (B.5) implies that the vector $(\beta,1)'$ can be written as $\Psi C_0'\phi$ divided by $[0,1]\Psi C_0'\phi$. Since the vector $(c_\beta, d_\beta)'$ in MM15 equals $C_0(\beta,1)'$, then:

$$||(c_\beta,d_\beta)'|| = ||C_0(\beta,1)'|| = ||C_0\Psi C_0'\phi||/|[0,1]\Psi C_0'\phi| = 1/|[0,1]\Psi C_0'\phi|.$$

Therefore,

$$1/||(c_\beta,d_\beta)'||^2 = ([0,1]\Psi C_0'\phi)^2.$$

This implies that

$$\sqrt{n}\pi \mid \phi \sim \mathcal{N}_k\left(\mathbf{0}\,,\ (||(c_\beta,d_\beta)'||^{-2}\Phi\right),$$

which is the same distribution as in MM15, up to a scaling constant. Also, MM15 assumes that the distribution of the angle of

$$C_0(\beta,1)'/||C_0(\beta,1)'||$$

is uniform on $[-\pi,\pi]$. Under (B.5) it follows that

$$C_0(\beta,1)'/||C_0(\beta,1)'|| = \phi,$$

where ϕ is uniformly distributed on the unit circle $\mathcal{S}^1$. Part ii) of exercise 5.2.4 in Stroock (1999) implies that ϕ can be written as $[\cos(\theta)', \sin(\theta)']'$ where θ is uniformly distributed on a connected interval of length $2pi$.

Distribution of $\sqrt{\lambda}(\beta-\beta_0)$ and λ: The Monte-Carlo exercises in Andrews et al. (2006) depend on the parameters:

$$\lambda \equiv n\pi\Phi^{-1}\pi, \text{ and } \sqrt{\lambda}(\beta-\beta_0).$$

Equation (B.6) implies that:

$$\begin{aligned}
\lambda &\equiv \left(([0,1]\Psi C_0'\phi)\Phi^{1/2}\rho\omega\right)\Phi^{-1}\left(([0,1]\Psi C_0'\phi)\Phi^{1/2}\rho\omega\right)\\
&= ([0,1]\Psi C_0'\phi)^2\rho^2\omega'\Phi^{1/2}\Phi^{-1}\Phi^{1/2}\omega\\
&= ([0,1]\Psi C_0'\phi)^2\rho^2.
\end{aligned}$$

Consequently, equation (B.5) implies that

$$\begin{aligned}\sqrt{\lambda}(\beta - \beta_0) &= \sqrt{([0,1]\Psi C_0'\phi)^2\rho^2}\left(\frac{(b_0'\Phi b_0)^{1/2}\phi_1}{[0,1]\Psi C_0'\phi}\right) \\ &= (b_0'\Phi b_0)^{1/2}\rho\phi_1.\end{aligned}$$

Therefore:

$$\sqrt{\lambda}(\beta - \beta_0) = \left(b_0'\Psi b_0\right)^{1/2}\rho\phi_1, \tag{B.7}$$

$$\lambda = ([0,1]\Psi C_0'\phi)^2\rho^2, \tag{B.8}$$

The probability density function of $(\sqrt{\lambda}(\beta - \beta_0), \lambda)$ is given in Figure 1 in the main text of the paper.

B.2. *Proof or Result 1:*

Preliminaries: The statistical model under consideration is:

$$\widehat{\gamma}_n \sim \mathcal{N}_{2k}\left(\begin{pmatrix} \beta\pi \\ \pi \end{pmatrix}, \Sigma/\sqrt{n} \right), \quad \text{where } \widehat{\gamma} \equiv \begin{pmatrix} (Z'Z)^{-1}Z'y \\ (Z'Z)^{-1}Z'x \end{pmatrix}. \tag{B.9}$$

By assumption

$$\Sigma = \Psi \otimes \Phi,$$

where Ψ is a matrix of dimension 2×2 and Φ is matrix of dimension $k \times k$ (both positive definite and symmetric). The model in (B.9) is transformed into a Gaussian location model with independent components by rotating the reduced-form parameters:

$$\begin{pmatrix} S_n \\ T_n \end{pmatrix} \equiv$$

$$\begin{bmatrix} ([b_0' \otimes \mathbb{I}_k)\Sigma(b_0 \otimes \mathbb{I}_k)]^{-1/2} & \mathbf{0} \\ \mathbf{0} & [(a_0' \otimes \mathbb{I}_k)\Sigma^{-1}(a_0 \otimes \mathbb{I}_k)]^{-1/2} \end{bmatrix} \begin{bmatrix} (b_0' \otimes \mathbb{I}_k) \\ (a_0' \otimes \mathbb{I}_k)\Sigma^{-1} \end{bmatrix} \sqrt{n}\widehat{\gamma}_n.$$

In the Kronecker case, this transformation can be written as:

$$\left(C_0 \otimes \Phi^{-1/2}\right)\sqrt{n}\widehat{\gamma}_n,$$

where

$$C_0 \equiv \begin{pmatrix} (b_0'\Psi b_0)^{-1/2} b_0' \\ \\ (a_0'\Psi^{-1}a_0)^{-1/2} a_0'\Psi^{-1} \end{pmatrix}, \quad b_0 = [1, -\beta_0]' \quad a_0 = [\beta_0, 1]'.$$

Consequently, the rotated statistical model becomes:

$$\begin{pmatrix} S_n \\ T_n \end{pmatrix} \sim \mathcal{N}_{2k}\left(\left(C_0 \otimes \Phi^{-1/2}\right)\sqrt{n} \begin{pmatrix} \beta\pi \\ \pi \end{pmatrix}, \ \mathbb{I}_{2k} \right) \tag{B.10}$$

In order to compute the WAP similar test, I need to integrate (B.10) with respect to the weights in (3.2). In the Kronecker case, these weights are given by:

$$\begin{pmatrix} \beta\pi \\ \pi \end{pmatrix} = n^{-1/2}\left(\Psi C_0' \otimes \Phi^{1/2}\right)\rho(\phi \otimes \omega),$$

where (ϕ, ω, ρ) are independent random variables with the following distributions:

$$\phi \sim \mathcal{U}(\mathcal{S}^1), \ \omega \sim \mathcal{U}(\mathcal{S}^k), \ \rho \sim \sqrt{\chi^2_k}.$$

Thus, integrating (B.10) with respect to the weights in (3.2) is equivalent to integrating the likelihood of:

$$\begin{pmatrix} S \\ T \end{pmatrix} \sim \mathcal{N}_{2k}\left(\rho\left(\phi \otimes \omega\right), \ \mathbb{I}_{2k}\right),$$

with respect to the weights for (ρ, ϕ, ω).

Derivation of the Integrated Likelihoods: Let $f(S, T; \rho, \phi, \omega)$ denote the Gaussian statistical model for (S, T) given parameters (ρ, ϕ, ω). This is:

$$f(S, T; \rho, \phi, \omega) = c_1 \exp\Big(-\frac{1}{2}([S', T']' - \rho(\phi \otimes \omega))'([S', T']' - \rho(\phi \otimes \omega))\Big),$$

where c_1 is a non-negative constant. The function $f(S,T;\rho,\phi,\omega)$ is thus analogous to $f(x;\theta)$ in Section 2 of the paper.

Step 1: (Integrate ω) Note that:

$$\begin{aligned}\widetilde{f}(S,T;\rho,\phi) &\equiv c_2 \int_{\mathcal{S}^{k-1}} f(S,T;\rho,\phi,\omega) d\lambda_{\mathcal{S}^{k-1}}(\omega) \\ &= a_2(Q)\exp\Big(-\rho^2/2\Big)\int_{\mathcal{S}^{k-1}} \exp\Big(([S,T]\phi)'\rho\omega\Big) d\lambda_{\mathcal{S}^{k-1}}(\omega)\end{aligned}$$

where $\lambda_{\mathcal{S}^{k-1}}(\cdot)$ is the uniform measure over the $k-1$ dimensional sphere $\mathcal{S}^{k-1}$ defined in Chamberlain (2007) and Stroock (1999). In addition,

$$a_2(Q) \equiv c_2 \exp\Big(-\frac{1}{2}[S'S+T'T]\Big)$$

c_2 is a non-negative constant.

Step 2: (Integrate ρ) By assumption $\rho \sim \sqrt{\chi^2_k}$ independently of ϕ and ω. The latter implies that the density of ρ is given by:

$$m_1(\rho) \equiv \frac{1}{2^{k/2}\Gamma(k/2)}(\rho^2)^{(k/2)-1}e^{-(\rho^2/2)}2\rho$$

Note that using Fubini's Theorem and the change of variables formula:

$$\int_{\mathbb{R}+} \widetilde{f}(S,T;\rho,\phi)m_1(\rho)d\rho$$

$$\begin{aligned}&= a_2(Q)\int_{\mathbb{R}+}\Big(\exp(-\rho^2/2)\int_{\mathcal{S}^{k-1}}\exp\Big(\Big[(S,T)\rho\phi\Big]'\boldsymbol{\omega}\Big)d\lambda_{\mathcal{S}^{k-1}}(\boldsymbol{\omega})\Big)m_1(\rho)d\rho \\ &= a_2(Q)\int_{\mathcal{S}^{k-1}}\Big(\int_{\mathbb{R}_+}\exp\Big(\Big[(S,T)\rho\phi\Big]'\boldsymbol{\omega}\Big)m_1(\rho)\exp(-\rho^2/2)d\rho\Big)d\lambda_{\mathcal{S}^{k-1}}(\boldsymbol{\omega}) \\ &= a_3(Q)\int_{\mathcal{S}^{k-1}}\Big(\int_{\mathbb{R}_+}\exp\Big(\Big[(S,T)\rho\phi\Big]'\boldsymbol{\omega}\Big)\exp(-\rho^2)\rho^{k-1}dp\Big)d\lambda_{\mathcal{S}^{k-1}}(\boldsymbol{\omega}) \\ &= a_3(Q)\int_{\mathcal{S}^{k-1}}\Big(\int_{\mathbb{R}_+}\exp\Big(\Big[(S,T)\phi\Big]'\rho\omega\Big)\exp(-(\rho\boldsymbol{\omega})'(\rho\boldsymbol{\omega}))\rho^{k-1}dp\Big)d\lambda_{\mathcal{S}^{k-1}}(\boldsymbol{\omega})\end{aligned}$$

where the last line follows from $\boldsymbol{\omega}'\boldsymbol{\omega}=1$ and $a_3(Q)=a_2(Q)2/(2^{k/2}\Gamma(k/2))$. Theorem 5.2.2, p. 86 in Stroock (1999) implies:

$$\begin{aligned}\int_{\mathbb{R}+}\widetilde{f}(S,T;\rho,\phi)m_1(\rho)d\rho &= a_3(Q)\int_{\mathbb{R}^K}\exp\Big(\Big[(S,T)\phi\Big]'x\Big)\exp(-x'x)dx \\ &\quad \text{(by applying Theorem 5.2.2 to the function } \exp\Big(\Big[(S,T)\phi\Big]'x\Big)\exp(-x'x)) \\ &= a_4(Q)\exp\Big(\frac{1}{4}\phi'Q\phi\Big),\ Q\equiv[S,T]'[S,T],\end{aligned}$$

where the last inequality follows by definition of the moment generating function of a k-dimensional multivariate normal evaluated at $(S,T)\phi$. Note that $a_4(Q)\equiv(2pi)^{k/2}a_3(Q)$.

Step 3: (Integrate ϕ) Part ii) of exercise 5.2.4 in p. 87 of Stroock (1999) implies that

$$\int_{\mathcal{S}^1} a_4(Q)\exp\Big(\frac{1}{4}\phi' Q\phi\Big)d\lambda_{\mathcal{S}^1}(\phi) = \frac{a_4(Q)}{2pi}\int_0^{2pi}\exp\Big(\frac{1}{4}[\cos(\theta),\sin(\theta)]Q[\cos(\theta),\sin(\theta)]'\Big)d\theta \equiv f^*(S,T).$$

Note that the largest and smallest eigenvalue of the matrix Q are given by:

$$\zeta_{max} = \frac{1}{2}\Big[(S'S+T'T)+\sqrt{(S'S-T'T)^2+4(S'T)^2}\Big]$$

$$\zeta_{min} = \frac{1}{2}\Big[(S'S+T'T)-\sqrt{(S'S-T'T)^2+4(S'T)^2}\Big]$$

The following step derives the numerator of the WAP-similar test in Result 1.

Step 3aux: We show that:

$$f^*(S,T) \quad = \quad a_4(Q)\exp\Big(\frac{1}{4}\Big(\zeta_{\max}+\zeta_{\min}\Big)\Big)I_0\Big(\frac{1}{4}(\zeta_{max}-\zeta_{min})\Big),$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind, defined in Abramowitz and Stegun (1964), Section 9.6, p. 375, and

$$a_4(Q) \propto (2pi)^{k/2}a_2(Q)2/(2^{k/2}\Gamma(k/2))\exp\Big(-\frac{1}{2}[S'S+T'T]\Big).$$

Proof: Let $L \equiv S'S - \zeta_{min}$. Note that L is the Likelihood Ratio Statistic as defined in Andrews et al. (2006) p. 722. The eigenvector associated to largest eigenvalue of the matrix Q equals:

$$e_{max} \equiv (L, S'T)'/\sqrt{L^2+(S'T)^2}$$

Define $\widehat{\theta} \in [0, 2pi]$ implicitly by the following equation:

$$[\cos(\widehat{\theta}),\sin(\widehat{\theta})]' = e_{max}$$

Therefore,

$$P \equiv \begin{pmatrix} \cos(\widehat{\theta}) & \sin(\widehat{\theta}) \\ \sin(\widehat{\theta}) & -\cos(\widehat{\theta}) \end{pmatrix}$$

yields the spectral decomposition of the matrix Q; that is:

$$P\begin{pmatrix} \zeta_{max} & 0 \\ 0 & \zeta_{min} \end{pmatrix}P' = Q.$$

Note that for any θ:

$$P'\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\widehat{\theta})\cos(\theta)+\sin(\widehat{\theta})\sin(\theta) \\ \sin(\widehat{\theta})\cos(\theta)-\cos(\widehat{\theta})\sin(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\widehat{\theta}-\theta) \\ \sin(\widehat{\theta}-\theta) \end{pmatrix}$$

Therefore:

$$
\begin{aligned}
f^*(S,T) &= \frac{a_4(Q)}{2pi}\int_0^{2pi} \exp\left(\frac{1}{4}\left[\zeta_{max}\cos^2(\widehat{\theta}-\theta)+\zeta_{min}\sin^2(\widehat{\theta}-\theta)\right]\right)d\theta \\
&= \frac{a_4(Q)}{2pi}\int_{\widehat{\theta}-2pi}^{\widehat{\theta}} \exp\left(\frac{1}{4}\left[\zeta_{max}\cos^2(\theta)+\zeta_{min}\sin^2(\theta)\right]\right)d\theta \\
&\qquad \text{(where he have changed the integration variable)} \\
&= \exp\left(\frac{1}{4}\zeta_{min}\right)\frac{a_4(Q)}{2pi}\int_{\widehat{\theta}-2pi}^{\widehat{\theta}} \exp\left(\frac{1}{4}\left[(\zeta_{max}-\zeta_{min})\cos^2(\theta)\right]\right)d\theta \\
&\qquad (\text{as} \sin^2(\theta)+\cos^2(\theta)=1) \\
&= \frac{a_4(Q)}{2pi}\exp\left(\frac{1}{4}\zeta_{min}\right)\exp\left(\frac{1}{8}(\zeta_{max}-\zeta_{min})\right) \\
&\quad \int_{\widehat{\theta}-2pi}^{\widehat{\theta}} \exp\left(\frac{1}{8}(\zeta_{max}-\zeta_{min})\cos(2\theta)\right)d\theta \\
&\qquad (\text{as } \cos^2(\theta)=(1+\cos(2\theta))/2) \\
&= \frac{a_4(Q)}{4pi}\exp\left(\frac{1}{4}\zeta_{min}\right)\exp\left(\frac{1}{8}(\zeta_{max}-\zeta_{min})\right) \\
x &\quad \int_{2(\widehat{\theta}-2pi)}^{2\widehat{\theta}} \exp\left(\kappa(Q)\cos(\theta)\right)d\theta \\
&\qquad \text{(where we have used the change of variable } \tilde{\theta}=2\theta) \\
&\qquad (\kappa(Q)\equiv\frac{1}{8}(\zeta_{max}-\zeta_{min})) \\
&= a_4(Q)\exp\left(\frac{1}{4}\zeta_{min}\right)\exp\left(\frac{1}{8}(\zeta_{max}-\zeta_{min})\right) \\
&\quad \frac{1}{2pi}\int_0^{2pi} \exp\left(\kappa(Q)\cos(u)\right)du \\
&\qquad \text{(where we have used the change of variable } u=(\widehat{\theta})-(\theta/2)
\end{aligned}
$$

Using the definition of the Von-Mises distribution and equation 3.5.18 in p. 36 of Mardia and Jupp (2000) it follows that:

$$
\begin{aligned}
f^*(S,T) &= a_4(Q)\exp\left(\frac{1}{4}\zeta_{min}\right)\exp\left(\frac{1}{8}(\zeta_{max}-\zeta_{min})\right)I_0\left(\kappa(Q)\right), \\
&= a_4(Q)\exp\left(\frac{1}{8}\left(\zeta_{max}+\zeta_{min}\right)\right)I_0\left(\frac{1}{8}(\zeta_{max}-\zeta_{min})\right)
\end{aligned}
$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind, defined in Abramowitz and Stegun (1964), Section 9.6, p. 375 and

$$
a_4(Q) \propto (2pi)^{k/2}a_2(Q)2/(2^{k/2}\Gamma(k/2))\exp\left(-\frac{1}{2}[S'S+T'T]\right)
$$

Q.E.D.

Proof of Result 1: I now derive the WAP-similar test. From the definitions of ζ_{max} and ζ_{min}:

$$\frac{1}{8}\Big(\zeta_{max}+\zeta_{min}\Big)=\frac{1}{8}S'S+T'T, \quad \frac{1}{8}\Big(\zeta_{max}-\zeta_{min}\Big)=\frac{1}{8}\Big((S'S-T'T)^2+4(S'T)^2\Big)^{1/2}.$$

From Step 3 (aux) above it follows that the integrated likelihood for independent weights:

$$\phi\sim\mathcal{U}(\mathcal{S}^1)\quad \omega\sim\mathcal{U}(\mathcal{S}^k)\quad \rho\sim\sqrt{\chi^2_k}$$

is given by:

$$f^*(S,T)=cons_1\exp(-\frac{1}{2}[S'S+T'T])\exp\left(\frac{1}{8}\Big(S'S+T'T\Big)\right)I_0\left(\frac{1}{8}\Big((S'S-T'T)^2+4(S'T)^2\Big)^{1/2}\right)$$

Note that the denominator in the expression of the WAP-similar test is

$$f(S|T;\beta_0)=cons_2\exp\Big(-\frac{1}{2}S'S\Big).$$

Consequently:

$$z_{\text{WAP}}(S,T)=\frac{cons_1}{cons_2}\exp\Big(-\frac{1}{2}T'T\Big)\exp\Big(\frac{1}{8}[S'S+T'T]\Big)I_0\left(\frac{1}{8}\Big((S'S-T'T)^2+4(S'T)^2\Big)^{1/2}\right)$$

The quantile function $c(T,\alpha)$ is continuous in T and, therefore, measurable. So that WAP-similar test rejects if and only if the test statistic

$$S'S-T'T+8\ln\left[I_0\Big(\frac{1}{8}\Big((S'S-T'T)^2+4(S'T)^2\Big)^{1/2}\Big)\right]$$

is larger than the critical value function $c^*(T,\alpha)$, defined as the $1-\alpha$ quantile (conditional on T) of the expression above under the distribution $S\sim\mathcal{N}_k(\mathbf{0},\mathbb{I}_k)$. This test is equivalent to the one presented in Result 1.

Asymptotic equivalence between the test statistic in Result 1 and the CLR as $t't\to\infty$: The CLR statistic can be written as

$$\frac{1}{2}\Big((s's-t't)+8x(s,t)\Big),$$

where

$$\begin{aligned} x(s,t) &\equiv \frac{1}{8}\Big((s's-t't)^2+4(s't)^2\Big)^{1/2},\\ &= \frac{t't}{8}\left(\left(\frac{s's}{t't}-1\right)^2+4\left(\frac{s't}{\sqrt{t't}}\right)^2\frac{1}{t't}\right)^{1/2}. \end{aligned}$$

The equation above implies $x(s,t)\to\infty$ as $t't\to\infty$ for any s. Moreover, the test statistic $z_{\text{WAP}}(s,t)$ in Result 1 equals

$$(s's-t't)+8\ln\left(I_0\Big(x(s,t)\Big)\right)+4\ln(2pi)+4\ln((1/8)t't).$$

The asymptotic approximation for the modified Bessel function $I_0(x)$ given in Olver (1997), p. 435 implies that

$$I_0(x)\Bigg/\frac{e^x}{(2pix)^{1/2}}\to 1,\ \text{as } x\to\infty,$$

Therefore, as $t't \to \infty$

$$\begin{aligned} z_{\text{WAP}}(s,t) &= (s's - t't) + 8\ln\left(\frac{e^{x(s,t)}}{(2pix(s,t))^{1/2}}\right) + 4\ln(2pi) + 4\ln((1/8)t't + o(1) \\ &= (s's - t't) + 8x(s,t) - 4\ln(2pix(s,t)) + 4\ln(2pi) + 4\ln((1/8)t't + o(1) \\ &= 2\text{CLR} - 4\ln(x(s,t)/(t't/8)) + o(1) \\ &= 2\text{CLR} + o(1). \end{aligned}$$